ACAP
Alabama Comprehensive Assessment Program

SUMMATIVE

Grade 8
Mathematics

Alabama Educator Instructional Supports

Alabama Course of Study Standards

DATA RECOGNITION
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CORPORATION

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Introduction

The *Alabama Educator Instructional Supports: Mathematics* is a companion to the 2019 *Alabama Course of Study: Mathematics* for Grades K–12. Instructional supports are foundational tools that educators may use to help students become independent learners as they build toward mastery of the *Alabama Course of Study* content standards. **Instructional supports are designed to help educators engage their students in exploring, explaining, and expanding their understanding of the content standards.**

The content standards contained within the course of study may be accessed on the Alabama State Department of Education (ALSDE) website: https://www.alabamaachieves.org/. When examining these instructional supports, educators are reminded that content standards indicate minimum content—what all students should know and be able to do by the end of each grade level or course. Local school systems may have additional instructional or achievement expectations and may provide instructional guidelines that address content sequence, review, and remediation.

The instructional supports are organized by standard. Each standard's instructional support includes a statement of the content standard, guiding questions with connections to mathematical practices, key academic terms, and additional resources.

Content Standards

The content standards are the statements from the 2019 *Alabama Course of Study: Mathematics* that define what all students should know and be able to do at the conclusion of a given grade level or course. Content standards contain minimum required content and complete the phrase "Students will ______."

Guiding Questions with Connections to Mathematical Practices

Guiding questions are designed to create a framework for the given standards and to engage students in exploring, explaining, and expanding their understanding of the content standards provided in the 2019 *Alabama Course of Study: Mathematics*. Therefore, each guiding question is written to help educators convey important concepts within the standard. By utilizing guiding questions, educators are engaging students in investigating, analyzing, and demonstrating knowledge of the underlying concepts reflected in the standard. An emphasis is placed on the integration of the eight Student Mathematical Practices.

The Student Mathematical Practices describe the varieties of expertise that mathematics educators at all levels should seek to develop in their students. They are based on the National Council of Teachers of Mathematics process standards and the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up: Helping Children Learn Mathematics*.

The Student Mathematical Practices are the same for all grade levels and are listed below.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Each guiding question includes a representative set of sample activities and examples that can be used in the classroom. The set of activities and examples is not intended to include all the activities and examples that would be relevant to the standard.

Key Academic Terms

These academic terms are derived from the standards and are to be incorporated into instruction by the educator and used by the students.

Additional Resources

Additional resources are included that are aligned to the standard and may provide additional instructional support to help students build toward mastery of the designated standard. Please note that while every effort has been made to ensure all hyperlinks are working at the time of publication, web-based resources are impermanent and may be deleted, moved, or archived by the information owners at any time and without notice. Registration is not required to access the materials aligned to the specified standard. Some resources offer access to additional materials by asking educators to complete a registration. While the resources are publicly available, some websites may be blocked due to Internet restrictions put in place by a facility. Each facility's technology coordinator can assist educators in accessing any blocked content. Sites that use Adobe Flash may be difficult to access after December 31, 2020, unless users download additional programs that allow them to open SWF files outside their browsers.

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Number Systems and Operations
Understand that the real number system is composed of rational and irrational numbers.
1. Define the real number system as composed of rational and irrational numbers. a. Explain that every number has a decimal expansion; for rational numbers, the decimal expansion repeats or terminates.

Guiding Questions with Connections to Mathematical Practices:

How can a number be identified as rational or irrational?

M.P.2. Reason abstractly and quantitatively. Identify a number as irrational or rational based on whether it can be written as a fraction. For example, $\sqrt{5}$ is irrational because it can't be written as a fraction, and 0.375 is rational because it can be written as the fraction $\frac{3}{8}$. Additionally, know that rational numbers include all integers, whole numbers, and natural numbers, while irrational numbers exclude all integers, whole numbers, and natural numbers.

- Ask students to identify which numbers are irrational given a list of numbers. For example, given the list 9, 4, $\sqrt{9}$, $\sqrt{4}$, $\sqrt{94}$, 94, −94, and $-\sqrt{94}$, identify $\sqrt{94}$ and $-\sqrt{94}$ as irrational.
- Ask students to identify all labels that apply to a number from the following list of labels: natural, whole, integer, rational, and irrational. For example, $\sqrt{17}$ can be labeled as an irrational number, and −4 can be labeled as an integer and as a rational number.

How do the decimal expansions of rational numbers compare to the decimal expansions of irrational numbers?

M.P.6. Attend to precision. Observe that rational numbers either terminate or have repeating digits, and know that irrational numbers are nonrepeating and nonterminating and so cannot be expressed as fractions. For example, the fraction $\frac{4}{9}$ can be expressed as a decimal by computing $4 \div 9$, which gives $0.\overline{4}$. The irrational number $\sqrt{2}$ written as a decimal is 1.4142 . . . with no repeating pattern or termination, so it cannot be expressed as a fraction unless it is rounded. Additionally, observe that the decimal expansion of rational numbers with a denominator that is not a product of 2's and 5's is nonterminating but has repeating digits.

- Provide students with three numbers that do not terminate and ask them to determine why one is rational and the other is irrational. For example, given the numbers 5.425425425 . . ., 5.477225575 . . ., and 5.44252525 . . . notice that 425 repeats in the first number and the first number can therefore be written as a fraction (i.e., $\frac{5{,}420}{999}$). As such, it is rational. In contrast, there is no discernable repeating pattern in the number 5.477225575 . . . . As such, it cannot be written as a fraction and is irrational. The third number does not repeat immediately but does eventually exhibit a repeating pattern and is therefore also rational.

- Provide students with a list of numbers and ask them to determine, without making any calculations, whether the decimal expansion of each number is terminating or nonterminating. If the decimal expansion is nonterminating, indicate whether it is repeating or nonrepeating. For example, given the numbers $\frac{8}{9}$, $\frac{9}{8}$, and $\sqrt{99}$, determine that the decimal expansion of $\frac{8}{9}$ is nonterminating and repeating, the decimal expansion of $\frac{9}{8}$ is terminating, and the decimal expansion of $\sqrt{99}$ is nonterminating and nonrepeating.

Key Academic Terms:

rational, irrational, decimal, fraction, repeating, terminating, approximate, decimal expansion, square root, cube root

Additional Resources:

- Lesson: Rational or irrational
- Lesson: Irrational numbers

Number Systems and Operations
Understand that the real number system is composed of rational and irrational numbers.
1. Define the real number system as composed of rational and irrational numbers. b. Convert a decimal expansion that repeats into a rational number.

Guiding Questions with Connections to Mathematical Practices:

How can equations be used to convert a decimal number with repeating digits into a fraction?

M.P.2. Reason abstractly and quantitatively. Know that a decimal with a single repeating digit can be represented as a fraction by first creating an equation with a variable on one side and the repeating decimal on the other side. Algebraic properties can then be used to represent the decimal as a fraction. First, create an equivalent equation by multiplying both sides of the equation by a power of 10 equal to the number of repeating decimal places. Next, subtract the first equation from the second equation. The difference of the two equations no longer includes the repeating decimal. The variable can be solved and expressed as a fraction. For example, begin with the decimal $0.\overline{5}$. Set the decimal equal to x such that $x = 0.555\ldots$. Then create an equivalent equation by multiplying both sides by 10 such that $10x = 5.555\ldots$. When $x = 0.555\ldots$ is subtracted from $10x = 5.555\ldots$, the result is $9x = 5$. After dividing both sides of the equation by 9, the result is $x = \frac{5}{9}$. Additionally, use equations to convert decimals with a sequence of two or more different digits that repeat by multiplying by other powers of 10.

- Provide students with a decimal number that has two or more repeating digits. Ask them to convert it to a fraction by setting up an equation and solving it. For example, provide students with the repeating decimal $0.\overline{78}$. In this case, students can first set the decimal equal to x.

$$x = 0.\overline{78}$$

Second, students can create an equivalent equation by multiplying both sides by 10^2, or 100.

$$100x = 78.\overline{78}$$

Third, the initial equation can be subtracted from the second equation, which will eliminate the repeating decimal.

$$99x = 78$$

Finally, both sides of the equation can be divided by 99 to create the fraction $\frac{78}{99}$, which is equivalent to $\frac{26}{33}$. Since $\frac{26}{33}$ and $0.\overline{78}$ are both equal to x, $\frac{26}{33}$ must be equal to $0.\overline{78}$.

- Provide students with a variety of decimal numbers that repeat. Ask them to determine the denominator when each number is converted to a fraction. For example, provide students with the following decimal numbers:
  - $0.\overline{08}$
  - $0.2\overline{75}$
  - $0.\overline{98765}$

When the first number is converted to a fraction, the denominator is 99 (i.e., $\frac{8}{99}$). When the second decimal is converted to a fraction, the denominator is 990 (i.e., $\frac{273}{990}$). When the third decimal is converted to a fraction, the denominator is 99,999 (i.e., $\frac{98,765}{99,999}$).

How can long division be used to verify that a decimal number with repeating digits was correctly converted to a fraction?

M.P.7. Look for and make use of structure. Know that if a decimal number with a single repeating digit has been correctly converted to a fraction, then dividing the numerator by the denominator using the standard algorithm will produce the original decimal. For example, if the decimal $0.\overline{7}$ is converted to the fraction $\frac{7}{9}$, the answer can be verified by solving the division problem $7 \div 9$. In this case, the digit 7 repeats infinitely after the decimal point in the quotient. Additionally, use long division to verify the accuracy of decimals with a sequence of two or more digits that repeat that have been converted into a fraction.

- Ask students to convert a multi-digit repeating decimal to a fraction and then verify the answer by finding the quotient of the numerator and the denominator with long division. For example, provide students with the number $0.\overline{63}$. After initially setting up the equation $x = 0.\overline{63}$, students can determine that $0.\overline{63}$ is equivalent to $\frac{7}{11}$. The conversion can be verified by solving the problem $7 \div 11$ with long division, as shown. Other long division strategies can also be used.

$$
\begin{array}{r}
0.6363\ldots \\
11 \overline{\big)\,7.0} \\
\underline{-\,66} \\
40 \\
\underline{-\,33} \\
70 \\
\underline{-\,66} \\
40 \\
\underline{-\,33} \\
7\ldots
\end{array}
$$

- Provide students with three suggested conversions of a repeating decimal to a fraction. Ask them to determine the correct conversion using long division. For example, provide students with the fractions $\frac{2}{99}$, $\frac{2}{9}$, and $\frac{9}{2}$. Ask them to demonstrate with long division which fraction is equivalent to $0.\overline{2}$. The first fraction is not equivalent, as shown.

$$
\begin{array}{r}
.0202\ldots \\
99\overline{)2.00} \\
-198 \\
\hline
20 \\
-\ \ 0 \\
\hline
200 \\
-198 \\
\hline
2\ldots
\end{array}
$$

Similarly, the third fraction is also not equivalent.

$$
\begin{array}{r}
4.5 \\
2\overline{)9.0} \\
-8 \\
\hline
10 \\
-10 \\
\hline
0
\end{array}
$$

Only the second fraction, $\frac{2}{9}$, provides the decimal expansion $0.\overline{2}$ when the problem $2 \div 9$ is solved with long division.

$$
\begin{array}{r}
.22\ldots \\
9\overline{)2.0} \\
-18 \\
\hline
20 \\
-18 \\
\hline
2\ldots
\end{array}
$$

Key Academic Terms:

rational, irrational, decimal, fraction, repeating, terminating, approximate, decimal expansion, square root, cube root

Additional Resources:

- Lesson: Rational or irrational
- Lesson: Irrational numbers

Number Systems and Operations
Understand that the real number system is composed of rational and irrational numbers.
2. Locate rational approximations of irrational numbers on a number line, compare their sizes, and estimate the values of the irrational numbers.

Guiding Questions with Connections to Mathematical Practices:

How does knowledge of perfect squares of rational numbers help determine the size of irrational numbers?

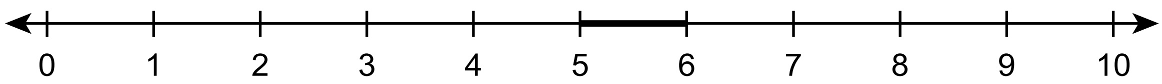
M.P.2. Reason abstractly and quantitatively. Estimate the value of an irrational number by using the closest rational known value or values. For example, $\sqrt{13}$ can be approximated by first finding the perfect squares 13 is between, which are 9 and 16. Since $\sqrt{9} = 3$ and $\sqrt{16} = 4$, $\sqrt{13}$ must be between 3 and 4. Additionally, $\sqrt{13}$ must be closer to 4 than to 3 since 13 is closer to 16 than to 9 on a number line. Further, approximate the value of an irrational number to the tenths place.

- Provide students with two consecutive whole numbers and ask them to determine an irrational number between them. For example, begin with the numbers 4 and 5. If an irrational number n is between 4 and 5, then the inequality $4 < n < 5$ can be used to represent the value of n. This means that $4^2 < n^2 < 5^2$, which can also be written as $16 < n^2 < 25$. Students can now identify a variety of irrational values for n that make the inequality true, including $\sqrt{17}$, $\sqrt{18}$, $\sqrt{19}$, $\sqrt{20}$, $\sqrt{21}$, $\sqrt{22}$, $\sqrt{23}$, and $\sqrt{24}$.

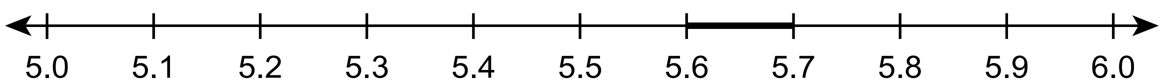
- Provide students with an irrational number and ask them to approximate its value to the nearest tenth. For example, begin with the irrational number $\sqrt{80}$. First, determine the two consecutive integers that $\sqrt{80}$ is between using perfect squares. In this case, $8 < \sqrt{80} < 9$ because $64 < 80 < 81$. As such, the ones value of $\sqrt{80}$ must be 8. There are now nine possible values for the tenths place: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, and 8.9. Given that 80 is much closer to 81 than 64, it is reasonable to use trial and error to see if 9 is the correct digit for the tenths place. Since $8.9^2 < 80 < 9^2$ is the same as $79.21 < 80 < 81$ and 80 is closer to 79.21 than it is to 81, conclude that 8.9 is an approximate value for $\sqrt{80}$ to the tenths place.

M.P.7. Look for and make use of structure. Estimate the value of an expression by locating the irrational number expression on a number line and using the closest rational value. For example, to approximate π^2 on a number line, locate π on a number line between 3 and 4. That means that π^2 will be between 9 and 16 because 3^2 is 9 and 4^2 is 16. To the nearest tenth, π is between 3.1 and 3.2, which means that π^2 is between 9.61 and 10.24. Taking it another step, π is between 3.14 and 3.15, which means that π^2 is between 9.8596 and 9.9225. As the pattern continues, the range for the value of π^2 gets smaller and becomes a better estimate of the actual value. Additionally, use previous knowledge of opposites to determine the approximate location on a number line of negative irrational numbers.

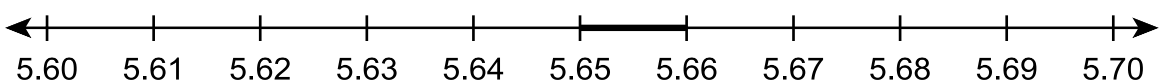
- Ask students to estimate the value of an irrational expression by first representing the location of an irrational number as an interval between two consecutive whole numbers on a number line. Then, ask students to represent the location of the number as an interval between two consecutive tenths on a number line. Finally, ask students to use intervals to create increasingly precise estimates of the value of the original irrational expression. For example, begin with $\sqrt{32} + 4$. The location of $\sqrt{32}$ must be between 5 and 6 on a number line because $5^2 < 32 < 6^2$.



Therefore, the value of $\sqrt{32} + 4$ must be between $5 + 4 = 9$ and $6 + 4 = 10$. When considering the tenths place, the location of $\sqrt{32}$ must be between 5.6 and 5.7 because $(5.6)^2 < 32 < (5.7)^2$.

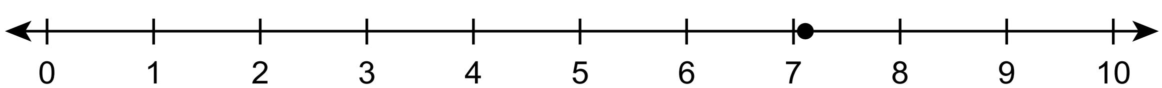


Therefore, the value of $\sqrt{32} + 4$ must be between $5.6 + 4 = 9.6$ and $5.7 + 4 = 9.7$. When considering the hundredths place, the location of $\sqrt{32}$ must be between 5.65 and 5.66 because $(5.65)^2 < 32 < (5.66)^2$.

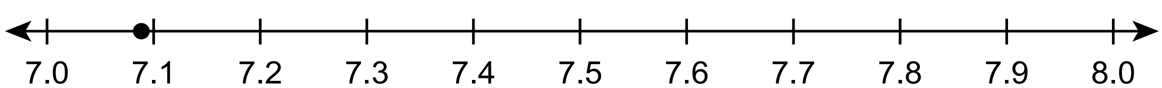


Therefore, the value of $\sqrt{32} + 4$ must be between $5.65 + 4 = 9.65$ and $5.66 + 4 = 9.66$.

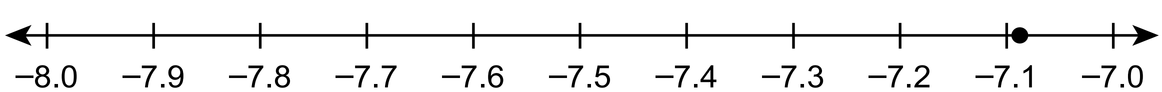
- Ask students to plot a negative irrational number in its approximate location on a number line to the tenths place. For example, provide students with the number $-\sqrt{50}$. Note that it is the same distance from 0 as $\sqrt{50}$ because $\sqrt{50}$ and $-\sqrt{50}$ are opposites. Then, ask students to determine two consecutive whole numbers that $\sqrt{50}$ is between. In this case, $\sqrt{50}$ is between 7 and 8 because $7^2 < 50 < 8^2$.



In considering the tenths place, $\sqrt{50}$ is between 7.0 and 7.1 because $(7.0)^2 < 50 < (7.1)^2$.



Finally, plot a point that represents $-\sqrt{50}$ by plotting a point that is the same distance from 0 as $\sqrt{50}$.



How do rational approximations of irrational numbers help compare the size of irrational numbers?

M.P.5. Use appropriate tools strategically. Compare the values of irrational numbers by using a calculator to find approximate numbers in decimal form. For example, $\sqrt{8} \approx 2.83$ and $\sqrt{10} \approx 3.16$. Since 2.83 is less than 3.16, $\sqrt{8} < \sqrt{10}$. Additionally, use rational approximations of irrational numbers to make comparisons in real-world contexts.

- Ask students to order a list of rational and irrational numbers from least to greatest using approximations. For example, given the numbers π, $\frac{10}{3}$, $3\frac{1}{8}$, and $\sqrt{11}$, identify the order from least to greatest as $3\frac{1}{8}$, π, $\sqrt{11}$, $\frac{10}{3}$.
- Ask students to solve a real-world problem involving irrational numbers by using rational approximations. For example, each square tile in Nic's bathroom has an area of 8 square inches. Each square tile in his kitchen has an area of 73 square inches. How many times as long is the side of a kitchen tile compared to the side of a bathroom tile? The side length of each tile can be determined by taking the square root of each area. After determining that the side length of a kitchen tile is approximately 8.54 inches ($\sqrt{73}$) and the side length of a bathroom tile is approximately 2.82 inches ($\sqrt{8}$), conclude that the length of a kitchen tile is about three times as long as the length of a bathroom tile.

Key Academic Terms:

rational, irrational, number line, approximate, estimate, square root, cube root

Additional Resources:

- Lesson: Irrational (and other!) numbers on the number line
- Lesson: Irrational numbers

Algebra and Functions
Apply concepts of integer exponents and radicals.
3. Develop and apply properties of integer exponents to generate equivalent numerical and algebraic expressions.

Guiding Questions with Connections to Mathematical Practices:

How can the properties of integer exponents be used to generate equivalent numerical expressions?

M.P.3. Construct viable arguments and critique the reasoning of others. Decompose and compose expressions that can be written in the form $a^m a^n = a^{m+n}$ to create equivalent expressions using exponents. When two terms being multiplied have the same base, the decomposed terms will show that the exponents can be added to find an equivalent expression. For example, $(-3)^5 \times (-3)^2$ decomposed is $(-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3)$, which is equivalent to $(-3)^7$. Therefore, to represent $(-3)^5 \times (-3)^2$, the exponents of 5 and 2 can be added: $(-3)^5 \times (-3)^2 = (-3)^{5+2} = (-3)^7$. Additionally, by decomposing several exponential expressions, use inductive reasoning to conclude that for a product of two exponential expressions with the same base, $a^m a^n$, the expression is equivalent to a^{m+n}.

- Ask students to decompose an exponential expression containing a product of two exponential expressions with the same base. Then, ask students to find an equivalent expression with a single base and a single exponent. For example, given the expression $4^3 \times 4^2$, write the equivalent expression $(4 \times 4 \times 4) \times (4 \times 4)$. Then, ask students to write the equivalent expression with a single base and a single exponent, 4^5, using the definition of exponents.

- Ask students to use inductive reasoning to generate a rule that can be used for any product of two exponential expressions of the same base, $a^m a^n$. For example, the following table has some information given, and the rest was completed by students.

Product of Same Bases	Decomposed Form	Single Base
$2^3 \times 2^4$	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	2^7
$(-5)^2 \times (-5)^4$	$(-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5)$	$(-5)^6$
$(-2)^3 \times (-2)^5$	$(-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2)$	$(-2)^8$
$7^5 \times 7$	$7 \times 7 \times 7 \times 7 \times 7 \times 7$	7^6

Generate the property $a^m a^n = a^{m+n}$ using inductive reasoning based on the results of the table.

- Ask students to find equivalent exponential expressions containing products of the same base with integer exponents. For example, write the expression $x^3 \bullet x^{-5} \bullet x^8$ as the equivalent expression x^6.

M.P.3. Construct viable arguments and critique the reasoning of others. Decompose and compose expressions that can be written in the form $(a^m)^n = a^{mn}$ to create equivalent expressions using exponents. For example, 5^6 decomposed is $5 \times 5 \times 5 \times 5 \times 5 \times 5$ and can be regrouped as $(5 \times 5 \times 5) \times (5 \times 5 \times 5)$ or $5^3 \times 5^3$, which is equivalent to $(5^3)^2$, or $(5^3)^2 = 5^{3 \bullet 2} = 5^6$. Additionally, by decomposing several exponential expressions of the form $(a^m)^n$, use inductive reasoning to conclude that for an exponential expression that is raised to a power, $(a^m)^n$, an equivalent expression is a^{mn}.

- Ask students to decompose an exponential expression that is raised to a power. Then, ask students to find an equivalent expression with a single exponent. For example, given the expression $(8^2)^5$, write the equivalent expressions $(8^2) \times (8^2) \times (8^2) \times (8^2) \times (8^2)$ and then $(8 \times 8) \times (8 \times 8) \times (8 \times 8) \times (8 \times 8) \times (8 \times 8)$. Finally, ask students to write the expression in exponential form with a single exponent, 8^{10}, using the definition of exponents.

- Ask students to use inductive reasoning to generate a rule that can be used for an exponential expression raised to a power, $(a^m)^n$. For example, the following table has some information given, and the rest was completed by students.

Exponential Expression Raised to a Power	Decomposed Form	Single Exponent
$(9^3)^2$	$9^3 \times 9^3$ or $9 \times 9 \times 9 \times 9 \times 9 \times 9$	9^6
$(4^2)^5$	$4^2 \times 4^2 \times 4^2 \times 4^2 \times 4^2$ or $4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$	4^{10}
$((-7)^4)^2$	$(-7)^4 \times (-7)^4$ or $(-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7)$	$(-7)^8$
$(a^4)^3$	$a^4 \times a^4 \times a^4$ or $a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a \times a$	a^{12}

Generate the property $(a^m)^n = a^{mn}$ using inductive reasoning based on the results of the table.

- Ask students to find equivalent exponential expressions raised to a power with a single exponent using the property generated. For example, write the expression $(x^6)^8$ as the equivalent expression x^{48}.

M.P.3. Construct viable arguments and critique the reasoning of others. Decompose and compose expressions that can be written in the form $(ab)^n = a^n b^n$ to create equivalent expressions using exponents. When terms are being multiplied together, it is helpful to remember the properties of multiplication. For example, $(ab)^4$ decomposed is $(a \times b) \times (a \times b) \times (a \times b) \times (a \times b)$, which is equivalent to $(a \times a \times a \times a) \times (b \times b \times b \times b)$ and can also be written as $a^4 b^4$. Additionally, by decomposing several exponential expressions, use inductive reasoning to conclude that for a product raised to a power, $(ab)^n$, an equivalent expression is $a^n b^n$.

- Ask students to decompose an exponential expression containing a product raised to a power. Then, ask students to find an equivalent expression where each base is raised to an exponent. For example, given the expression $(6y)^3$, write the equivalent expression $6y \times 6y \times 6y$. Then, write the equivalent expression $6 \times 6 \times 6 \times y \times y \times y$. Finally, ask students to write an equivalent expression where each base is raised to an exponent, $6^3 y^3$, using the definition of exponents.

- Ask students to use inductive reasoning to generate a rule that can be used for any product raised to a power, $(ab)^n$. For example, the following table has some information given, and the rest was completed by students.

Product Raised to a Power	**Decomposed Form**	**Each Base Raised to Power**
$(-3n)^4$	$(-3n) \times (-3n) \times (-3n) \times (-3n)$ or $(-3) \times (-3) \times (-3) \times (-3) \times n \times n \times n \times n$	$(-3)^4 n^4$
$(2a)^5$	$(2a) \times (2a) \times (2a) \times (2a) \times (2a)$ or $2 \times 2 \times 2 \times 2 \times 2 \times a \times a \times a \times a \times a$	$2^5 a^5$
$(-7b)^3$	$(-7b) \times (-7b) \times (-7b)$ or $(-7) \times (-7) \times (-7) \times b \times b \times b$	$(-7)^3 b^3$
$(8m)^2$	$(8m) \times (8m)$ or $8 \times 8 \times m \times m$	$8^2 m^2$

Generate the property $(ab)^n = a^n b^n$ using inductive reasoning based on the results of the table.

- Ask students to find equivalent exponential expressions containing products raised to powers. For example, write the expression $(-5z^2)^4$ as the equivalent expression $(-5)^4 z^8$.

How are negative exponents used to represent repeated division?

M.P.7. Look for and make use of structure. Connect patterns of exponents to repeated multiplication and division. For example, $\frac{8^5}{8^3} = \frac{8 \bullet 8 \bullet 8 \bullet 8 \bullet 8}{8 \bullet 8 \bullet 8} = 8^2$ by decomposition and $\frac{8^5}{8^3} = 8^{5-3} = 8^2$ using the properties of exponents. The same patterns hold when the power of the denominator is greater than the power of the numerator: $\frac{8^3}{8^5} = \frac{8 \bullet 8 \bullet 8}{8 \bullet 8 \bullet 8 \bullet 8 \bullet 8} = \frac{1}{8^2}$ by decomposition and $\frac{8^3}{8^5} = 8^{3-5} = 8^{-2}$ using the properties of exponents. Therefore, the fact that $8^{-2} = \frac{1}{8^2} = \frac{1}{8 \bullet 8}$ means that a negative exponent can be interpreted to mean repeated division. Another approach is to decompose the expression $\frac{8^3}{8^5}$ into $\frac{8 \bullet 8 \bullet 8}{8 \bullet 8 \bullet 8 \bullet 8 \bullet 8}$ and then observe that $\frac{8}{8} \bullet \frac{8}{8} \bullet \frac{8}{8} \bullet \frac{1}{8} \bullet \frac{1}{8} = 1 \bullet 1 \bullet 1 \bullet \frac{1}{8} \bullet \frac{1}{8} = \frac{1}{8 \bullet 8} = \frac{1}{64}$. Additionally, when using decomposition on several exponential expressions containing a quotient of same bases, use inductive reasoning to conclude that for $\frac{a^m}{a^n}$, an equivalent expression with a single base and a positive power is a^{m-n} when m is greater than n, or is $\frac{1}{a^{n-m}}$ when n is greater than m. Further, demonstrate that as the exponent increases for each integer value of n in a^n, the value of the expression is multiplied by a factor of a, and as the exponent decreases for each integer value of n in a^n, the value of the expression is multiplied by $\frac{1}{a}$. Conclude that an exponential expression of the form a^{-n} has an equivalent expression of $\frac{1}{a^n}$.

- Ask students to use inductive reasoning to generate a rule that can be used for any quotient of two exponential expressions of the same base, $\frac{a^m}{a^n}$. For example, give students the expressions shown in the left column of the table and ask them to write the equivalent forms shown in the other columns of the table.

Quotient of Same Bases	Decomposed Form	Single Base
$\frac{5^9}{5^3}$	$\frac{5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5}{5 \bullet 5 \bullet 5}$	5^6
$\frac{2^8}{2^5}$	$\frac{2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2}{2 \bullet 2 \bullet 2 \bullet 2 \bullet 2}$	2^3
$\frac{3^4}{3^8}$	$\frac{3 \bullet 3 \bullet 3 \bullet 3}{3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3}$	$\frac{1}{3^4}$
$\frac{7^5}{7^{10}}$	$\frac{7 \bullet 7 \bullet 7 \bullet 7 \bullet 7}{7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7 \bullet 7}$	$\frac{1}{7^5}$

Use inductive reasoning to conclude that for an exponential expression in the form $\frac{a^m}{a^n}$, an equivalent expression with a single base and a positive power is a^{m-n} when m is greater than n or $\frac{1}{a^{n-m}}$ when n is greater than m.

- Ask students to generate a rule that can be used for exponential expressions written with a negative exponent, a^{-n}. For example, give students the expressions shown in the left column of the table and ask them to write the equivalent forms shown in the other column of the table.

Power of 2	Value
2^3	8
2^2	4
2^1	2
2^0	1
2^{-1}	$\frac{1}{2}$ or $\frac{1}{2^1}$
2^{-2}	$\frac{1}{4}$ or $\frac{1}{2^2}$
2^{-3}	$\frac{1}{8}$ or $\frac{1}{2^3}$

Notice that as the exponent decreases, the values are divided by 2 each time. Determine that an exponential expression of the form a^{-n} is equivalent to $\frac{1}{a^n}$, based on the table.

- Ask students to use the exponent addition properties to derive the values of expressions with negative exponents. For example, give students the expression $2^{-1} \times 2^2$. This expression is equal to 2^1. Ask students to use the facts that $2^2 = 4$ and $2^1 = 2$ to create the equation $2^{-1} \times 4 = 2$. The value of 2^{-1} is $\frac{1}{2}$.

Why is a nonzero rational number raised to the zero power equivalent to 1?

M.P.7. Look for and make use of structure. Demonstrate that as the exponent increases by 1, the base is multiplied by itself one more time, and as the exponent decreases by one, the base is divided by itself one more time. For example, $\frac{a^2}{a} = (a \bullet a) \bullet \frac{1}{a} = a^1$. Dividing by a reduced the power by 1, from a^2 to a^1. Continuing, $\frac{a}{a} = a \bullet \frac{1}{a} = 1$. The power reduces by 1, meaning a will be divided by a, so $a^0 = 1$. In addition, $\frac{a}{a} = 1$, so $\frac{a}{a} = a^{1-1} = a^0 = 1$. Further, observe that a must be nonzero because dividing by 0 is undefined.

- Ask students to consider a table containing whole number powers of a natural number and draw a conclusion about the value of an exponential expression with a power of 0. For example, the following table has some information given, and the rest was completed by students.

Power of 3	Value
3^3	27
3^2	9
3^1	3
3^0	1

Determine that as the value of the exponent increases by 1, the value of the expression is multiplied by a factor of 3. Also, determine that as the value of the exponent decreases by 1, the value of the expression is divided by 3. Conclude that $3^0 = 1$.

- Ask students to generate a property through inductive reasoning that can be used to write an expression equivalent to a^0 without an exponent. For example, the following table has some information given, and the rest was completed by students.

Powers of 4	Value	Powers of −2	Value	Powers of −3	Value	Powers of a	Value
4^3	64	$(-2)^3$	−8	$(-3)^3$	−27	a^3	a^3
4^2	16	$(-2)^2$	4	$(-3)^2$	9	a^2	a^2
4^1	4	$(-2)^1$	−2	$(-3)^1$	−3	a^1	a
4^0	1	$(-2)^0$	1	$(-3)^0$	1	a^0	1

Use inductive reasoning to determine that $a^0 = 1$ for any nonzero real number a.

Key Academic Terms:

integer, exponent, power, negative, base, reciprocal, inverse, equivalent, radical, factor of, properties of exponents

Additional Resources:

- Lesson: Properties of exponents
- Lesson: Understanding negative exponents

Algebra and Functions
Apply concepts of integer exponents and radicals.
4. Use square root and cube root symbols to represent solutions to equations. a. Evaluate square roots of perfect squares (less than or equal to 225) and cube roots of perfect cubes (less than or equal to 1000).

Guiding Questions with Connections to Mathematical Practices:

What is the relationship between the value of the square root of a number and the value after the number is raised to the power of 2?

M.P.2. Reason abstractly and quantitatively. Define the square root of x as a number that, when multiplied by itself, is equal to x, and know that x can be decomposed as $x = \sqrt{x} \bullet \sqrt{x}$. The exponent of 2 means to use the base as a factor 2 times, so x^2 can be decomposed as $x^2 = x \bullet x$. To find the solution to an equation such as $x^2 = p$, find a number that when multiplied by itself is p. That number is represented as $\sqrt{p}$. The radical symbol in $\sqrt{p}$ indicates the principal square root, and the answer is the positive solution to $x^2 = p$, such as $\sqrt{16} = 4$. The square and the square root are inverse operations. For example, $(\sqrt{25})^2 = 25$ and $\sqrt{25^2} = 25$. Additionally, observe that in the equation $x^2 = p$ when p is a positive number, the equation has two solutions because there are two values that can be substituted into the equation for x that make the equation true. Further, know that when an expression containing a radical symbol has a negative sign outside the radical symbol, the negative square root is being indicated, such as $-\sqrt{49} = -7$.

- Ask students to find solutions to the equation $x^2 = p$ when p is a perfect square. For example, give students the equation $x^2 = \frac{121}{81}$ to solve. Determine that the equation has two solutions, $\frac{11}{9}$ and $-\frac{11}{9}$ because $(\frac{11}{9})^2 = \frac{121}{81}$ and $(-\frac{11}{9})^2 = \frac{121}{81}$.

- Ask students to evaluate expressions containing small perfect squares. For example, give students the expressions shown in the left column of the table and ask them to find the values shown in the right column.

Expression	Value
$\sqrt{144}$	12
$\sqrt{\frac{9}{64}}$	$\frac{3}{8}$
$-\sqrt{25}$	-5
$-\sqrt{\frac{16}{49}}$	$-\frac{4}{7}$

What is the relationship between the value of a cube root of a number and the value after the number is raised to the power of 3?

M.P.2. Reason abstractly and quantitatively. Define the cube root of x as a number that, when used as a factor 3 times, has a product that is equal to x, and observe that x can be decomposed as $x = \sqrt[3]{x} \bullet \sqrt[3]{x} \bullet \sqrt[3]{x}$. The exponent of 3 means to use the base as a factor 3 times, so x^3 can be decomposed as $x^3 = x \bullet x \bullet x$. To find the solution to an equation such as $x^3 = p$, find a number that when used as a factor 3 times has a product equal to p. That number is represented as $\sqrt[3]{p}$. The cube and the cube root are inverse operations. For example, $(\sqrt[3]{125})^3 = 125$ and $\sqrt[3]{125^3} = 125$. Additionally, verify that equations of the form $x^3 = p$ have exactly one solution.

- Ask students to find solutions to $x^3 = p$ when p is a positive or negative perfect cube number. For example, give students the equation $x^3 = 27$. Determine that the solution to the equation is $x = 3$ because $3 \times 3 \times 3 = 27$. Additionally, the equation $x^3 = -125$ has the solution $x = -5$ because $(-5) \times (-5) \times (-5) = -125$.

- Ask students to evaluate expressions containing small perfect cubes. For example, give students the expressions shown in the left column of the table and ask them to find the values shown in the right column.

Expression	Value
$\sqrt[3]{64}$	4
$\sqrt[3]{8}$	2
$\sqrt[3]{-1}$	-1
$\sqrt[3]{-27}$	-3

- Ask students to evaluate expressions containing the cube root of a number cubed or a cube of the cube root of a number. For example, ask students to evaluate $\left(\sqrt[3]{-8}\right)^3$ by recognizing that $\sqrt[3]{-8}$ defines a number that, when used as a product three times, equals −8. Therefore, because $\left(\sqrt[3]{-8}\right)^3$ uses $\sqrt[3]{-8}$ as a product three times, it has a value of −8. Further, students can use the same reasoning to conclude that $\sqrt{4^2} = 4$.

Key Academic Terms:

rational, square root, cube root, exponent, irrational, power, perfect square, perfect cube, radical, principal square root

Additional Resources:

- Video: Real-life math | Police Accident Investigator
- Video: Real-life math | Radiologist
- Lesson: Understanding perfect cubes and cube roots
- Lesson: Irrational numbers
- Lesson: Cube root solutions
- Video: Approximating square roots of nonperfect squares

Algebra and Functions
Apply concepts of integer exponents and radicals.
4. Use square root and cube root symbols to represent solutions to equations. b. Explain that the square root of a non-perfect square is irrational.

Guiding Questions with Connections to Mathematical Practices:

When is the square root of a whole number irrational?

M.P.2. Reason abstractly and quantitatively. Observe that the square root of any whole number is rational only if the whole number is a perfect square. If the whole number is not a perfect square, then the square root of the number is irrational. For example, $\sqrt{81}$ is a rational number because 81 is a perfect square. By contrast, $\sqrt{91}$ is an irrational number because 91 is not a perfect square. There is no rational value of x such that x^2 is equal to 91. Additionally, use perfect squares to determine which two consecutive whole numbers an irrational square root is located between.

- Ask students to explain why $\sqrt{10}$ is irrational. Students can first identify that $\sqrt{10}$ is less than $\sqrt{16}$ and greater than $\sqrt{9}$, which are square roots of perfect squares of consecutive integers. Therefore, $\sqrt{10}$ is less than 4 and greater than 3. There are no whole numbers between 3 and 4, so $\sqrt{10}$ is either a rational number that is not a whole number or it is irrational. The decimal expansion of $\sqrt{10}$ is 3.16227766017 . . . . The decimal expansion does not terminate or repeat. Therefore, $\sqrt{10}$ is irrational.

- Ask students to determine whether a square root is rational or irrational. For example, give students a table like the one shown and have them write the correct term for each solution.

Square Root	Rational or Irrational
$\sqrt{2}$	irrational
$\sqrt{25}$	rational
$\sqrt{49}$	rational
$\sqrt{99}$	irrational

Key Academic Terms:

rational, square root, cube root, exponent, irrational, power, perfect square, principal square root

Additional Resources:

- Video: Real-life math | Police Accident Investigator
- Video: Real-life math | Radiologist
- Lesson: Understanding perfect cubes and cube roots
- Lesson: Irrational numbers
- Lesson: Cube root solutions
- Video: Approximating square roots of nonperfect squares

Algebra and Functions
Apply concepts of integer exponents and radicals.
5. Estimate and compare very large or very small numbers in scientific notation.

Guiding Questions with Connections to Mathematical Practices:

How can integer exponents of 10 be used to compare very large and very small quantities?

M.P.6. Attend to precision. Represent a very large or very small quantity by first rounding it to the nearest single significant digit and then multiplying that number by the appropriate power of 10, which is called scientific notation. For example, 0.00000000000000298 can be written in scientific notation as 3×10^{-15} and 0.0000000000008976 can be written as 9×10^{-13}. To compare the two numbers, 9 is three times 3, and the power of 10 is 10^2 as great in 9×10^{-13}, so 9×10^{-13} is 300 times as great as 3×10^{-15}. Additionally, remember that to determine the single digit, a number should always be rounded to its highest place value.

- Ask students to estimate a very large quantity in scientific notation. For example, estimate 38,400,000,000 to be 4×10^{10} because 38,400,000,000 rounded to its highest place value is 40,000,000,000. Use the leading digit of 4 as the single digit and determine that 4 needs to be multiplied by 10 ten times to create a value of 40,000,000,000.
- Ask students to estimate a very small quantity in scientific notation. For example, estimate 0.0000000613 to be 6×10^{-8} because 0.0000000613 rounded to its highest place value is 0.00000006. Use the leading digit of 6 as the single digit and determine that 6 needs to be divided by 10 eight times to create a value of 0.00000006. Observe that dividing by 10 eight times is equivalent to multiplying by 10^{-8}.
- Ask students to compare numbers that have been represented in scientific notation. For example, given 2×10^{-6} and 5×10^{-8}, determine that 2×10^{-6} is the greater of the two values. Then, determine that 2×10^{-6} is 40 times as great as 5×10^{-8} by first finding that 2 is equal to 0.4 times 5. Then, notice that the power of 10 in 2×10^{-6} is 10^2 times as great as the power of 10 in 5×10^{-8}. Since 0.4 times 10^2 is 40, that means that 2×10^{-6} is 40 times as great as 5×10^{-8}.

Key Academic Terms:

integer, power, exponent, significant digit, estimate, scientific notation, base, rounding, negative, properties of exponents

Additional Resources:

- Video: Scientific notation
- Lesson: Introduction to scientific notation
- Lesson: One quadrillion yen
- Activity: The Solar System, test tubes, and scientific notation

Algebra and Functions
Apply concepts of integer exponents and radicals.
6. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. a. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities.

Guiding Questions with Connections to Mathematical Practices:

When is it appropriate to use scientific notation?

M.P.2. Reason abstractly and quantitatively. Use scientific notation when the context calls for it and the quantities are either very large or very small. For example, use scientific notation as an estimate for the number of stars visible in the sky on a clear night, the number of cells on a microscope slide, or the mass of a planet but not for the amount of change in a pocket or the number of people on a bus. Additionally, generate examples of contexts that warrant the use of scientific notation and those that do not warrant the use of scientific notation.

- Ask students to explain why very large values in certain contexts warrant the use of scientific notation while other values in contexts involving the same units do not warrant the use of scientific notation. For example, determine that to express the distance, in miles, between a student's home and the student's school, the use of scientific notation is not warranted because the distance between a student's home and the student's school can be expressed using a relatively small number of digits. However, the distance between Earth and Mars, in miles, does warrant the use of scientific notation because the distance between Earth and Mars cannot be expressed using a relatively small number of digits.

- Ask students to explain why very small values in certain contexts warrant the use of scientific notation while other values in contexts involving the same units do not warrant the use of scientific notation. For example, determine that to express the length of a caterpillar, in millimeters, the use of scientific notation is not warranted but the diameter of a red blood cell in millimeters does warrant the use of scientific notation. The length of the caterpillar can be expressed using a relatively small number of digits but the diameter of the red blood cell cannot be expressed using a relatively small number of digits.

How can the properties of exponents be used to solve problems written in scientific notation?

M.P.6. Attend to precision. Connect knowledge of the properties of exponents to scientific notation to solve problems. For example, $(4.8 \times 10^{12}) \times (2.1 \times 10^{6})$ can be written as $(4.8 \times 2.1) \times (10^{12} \times 10^{6})$, which can be written as $10.08 \times 10^{12+6}$ or 10.08×10^{18}. The number 10.08×10^{18} represented in scientific notation is 1.008×10^{19}. Additionally, know that products and quotients of numbers expressed in scientific notation can be rearranged and regrouped using the commutative and associative properties.

- Ask students to find a product of two numbers expressed in scientific notation using the commutative and associative properties. For example, given the product $(3.1 \times 10^{8}) \times (4.2 \times 10^{7})$, use the commutative and associative properties to express the product as $(3.1 \times 4.2) \times (10^{8} \times 10^{7})$. Perform the multiplication in each set of parentheses to get a result of 13.02×10^{15}. Create an equivalent expression that is expressed in scientific notation by dividing the first factor of the product by 10 and multiplying the second factor of the product by 10 to get 1.302×10^{16}.

- Ask students to find a quotient of two numbers expressed in scientific notation using the commutative and associative properties. For example, given the quotient $(2.4 \times 10^{5}) \div (9.6 \times 10^{9})$, use the commutative and associative properties to express the quotient as $(2.4 \div 9.6) \times (10^{5} \div 10^{9})$. Perform the division in each set of parentheses to get a result of 0.25×10^{-4}. Create an equivalent expression that is expressed in scientific notation by multiplying the first factor of the product by 10 and dividing the second factor of the product by 10 to get 2.5×10^{-5}.

How is the appropriate unit determined when solving problems using scientific notation?

M.P.1. Make sense of problems and persevere in solving them. Determine units by making sense of the context. For example, when calculating long distances, use kilometers instead of centimeters, or when tracking levels of contaminants in a water system, use milliliters instead of liters. Additionally, know that the use of the appropriate unit within a given context allows for numbers to be expressed in a meaningful way with fewer digits.

- Ask students to determine an appropriate unit of measurement for a given context and explain why that unit of measurement is appropriate for the given context. For example, given the amount of water contained in all of Earth's oceans, use cubic miles instead of cubic feet to express the amount of water and explain that because the amount of water is very large, cubic miles is the more appropriate unit to use.

- Ask students to analyze statements made about the appropriate unit of measurement in a given situation and determine the validity of the statement. For example, give students the prompt "Sam tracked the speed at which he rode his bike in feet per hour. Determine a more appropriate unit that Sam could use to represent his speed and explain how you know the unit you chose is more appropriate."

  Sam did not use an appropriate unit because the speed in feet per hour would be a very large number. He should use either miles per hour or feet per second to measure his speed.

Key Academic Terms:

scientific notation, significant digit, decimal, exponent, power, unit, radical, base, operations, properties of exponents

Additional Resources:

- Video: Scientific notation
- Lesson: Operations with numbers in scientific notation
- Lesson: Perform operations with scientific notation 1
- Activity: Balance the scale

Algebra and Functions

Apply concepts of integer exponents and radicals.

6. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used.

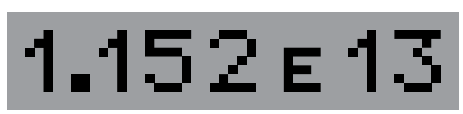
b. Interpret scientific notation that has been generated by technology.

Guiding Questions with Connections to Mathematical Practices:

How does technology represent numbers written in scientific notation?

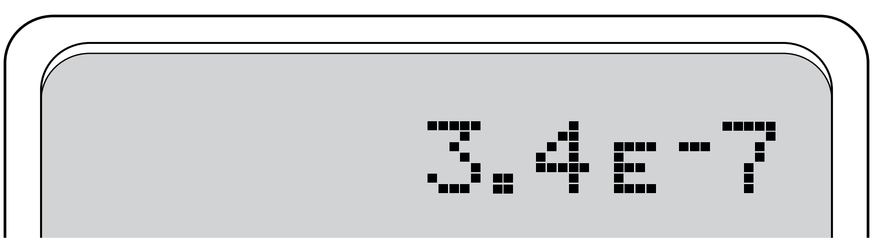
M.P.5. Use appropriate tools strategically. Analyze the result of a math problem that requires technology and has a solution that is either a very large or very small number. For example, a series of computations on a calculator gives 5.2E31 as an output. Know that this is an alternate representation for 5.2×10^{31}. Additionally, know how to input numbers expressed in scientific notation into a calculator including products and quotients of numbers that are expressed in scientific notation.

- Ask students to complete a series of computations using a calculator that result in the calculator giving a number expressed in scientific notation. Then ask students to interpret the output given by the calculator. For example, give students the product 2,400,000 × 4,800,000 to compute using a calculator. The calculator generates the output shown.



Students should know that the output is an alternate representation of 1.152×10^{13}.

- Ask students to compute a product or quotient of numbers expressed in scientific notation using a calculator and interpret the output given by the calculator. For example, give students the quotient $(8.5 \times 10^{7}) \div (2.5 \times 10^{14})$ to compute using a calculator. Students input the quotient into a calculator using parentheses around each number in scientific notation and the ^ button to represent exponents. The calculator generates the output shown.



Students should know that the output is an alternate representation of 3.4×10^{-7}.

Key Academic Terms:

scientific notation, significant digit, decimal, exponent, power, unit, radical

Additional Resources:

- Video: Scientific notation
- Lesson: Operations with numbers in scientific notation
- Lesson: Perform operations with scientific notation 1
- Activity: Balance the scale

Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
7. Determine whether a relationship between two variables is proportional or non-proportional.

Guiding Questions with Connections to Mathematical Practices:

How can a table be used to determine whether the relationship between two variables represented in an equation is proportional or non-proportional?

M.P.2. Reason abstractly and quantitatively. Know that an equation represents a proportional relationship if the ratio of the y-coordinate to the x-coordinate is the same for all coordinate pairs in a table. For example, a table of values that includes the pairs (1, 3), (2, 6), and (3, 9) can be created from the equation $y = 3x$. In this case, the relationship between x and y is proportional in all three pairs because each y-value is three times the value of its corresponding x value. Additionally, observe that an equation without a constant added to the x term represents a proportional relationship, while an equation with a constant added to the x term is non-proportional.

- Provide students with linear relationships shown as equations and corresponding tables of values. Ask them to explain whether the relationships are proportional or non-proportional. For example, provide students with the following equations and tables of values.

$y = 2x$

x	y
1	2
10	20
100	200
1,000	2,000

$y = \frac{1}{2}x + 2$

x	y
2	3
4	4
6	5
8	6

After analyzing the tables, students should be able to explain that the equation $y = 2x$ represents a proportional relationship because each y-value is double the corresponding x-value. By contrast, the equation $y = \frac{1}{2}x + 2$ represents a non-proportional relationship because the y-value of the first pair in the table is $1\frac{1}{2}$ times the corresponding x-value, while the y-value of the second pair in the table is equal to the corresponding x-value.

- Provide students with a relationship that is represented by an equation. Ask them to create a table of values to demonstrate whether the relationship is proportional or non-proportional. For example, provide students with the equation $y = \frac{2}{3}x$. Students can create a table of values to determine whether the ratio of the y-value to the x-value is the same for all corresponding pairs. Some possible values are shown in the table.

x	y
1	$\frac{2}{3}$
3	2
6	4

In this case, students should be able to conclude that the equation $y = \frac{2}{3}x$ represents a proportional relationship because every y-value is $\frac{2}{3}$ of the corresponding x-value.

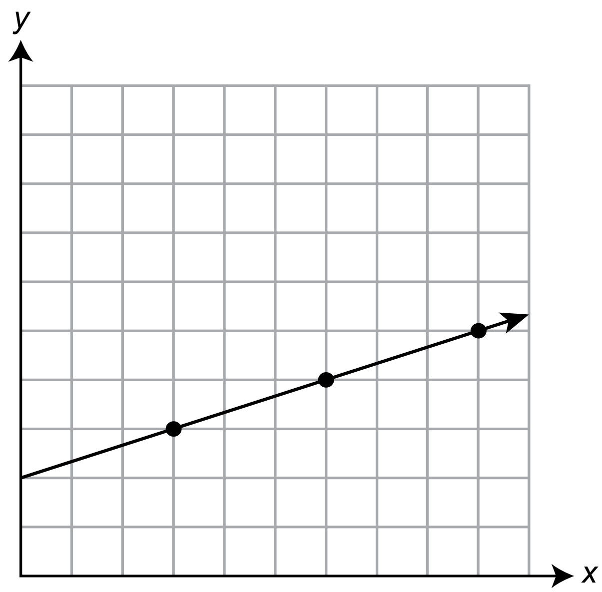
- Provide students with a variety of relationships that are represented by equations. Ask them to determine, by inspection, whether the relationships are proportional or non-proportional. For example, provide students with the following equations.
  - $y = \frac{1}{2}x$
  - $y = \frac{1}{2}x + 2$
  - $y = 4x$
  - $y = 4x + \frac{1}{2}$

In this case, the first and third equations represent proportional relationships because there is not a constant term. By contrast, the second and fourth equations represent non-proportional relationships because there is a constant term added.

How can the graph of a relationship between two variables be used to determine whether the relationship is proportional?

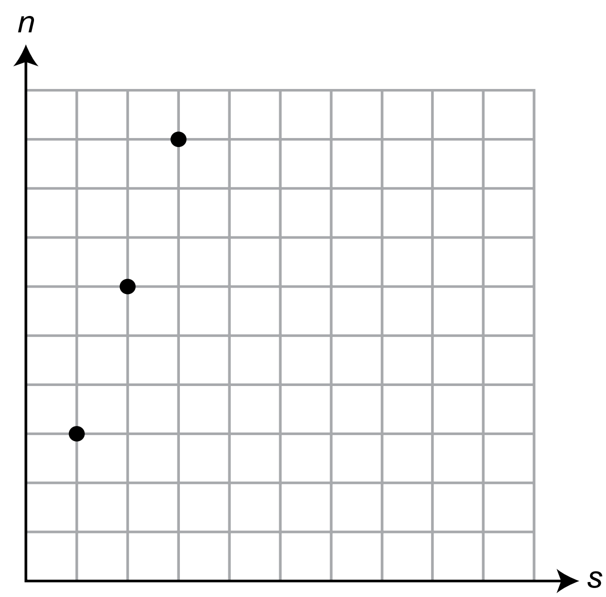
M.P.4. Model with mathematics. Know that if the graph of a relationship between two variables is a straight line through the origin, then the relationship is proportional. For example, it can be determined from the graph of $y = 2x + 1$ that the relationship is non-proportional because the line does not pass through the origin. When the value of x is 0, the value of y is 1 instead of 0. By contrast, it can be determined from the graph of $y = 2x$ that the relationship is proportional because the line does pass through the origin. When x is zero, the value of y is also 0. Additionally, distinguish between real-world relationships that are proportional and those that are linear but non-proportional.

- Provide students with the graph of a linear relationship. Ask them to explain why the graph represents either a proportional relationship or a non-proportional relationship. For example, provide students with the following graph.



In this case, students should be able to explain that the graph represents a non-proportional relationship because the line does not pass through the origin. Alternatively, students could explain that the graph represents a non-proportional relationship because the proportions of the y-values to the x-values are not the same for all ordered pairs on the graph. More specifically, the y-value is the same as the x-value at the point (3, 3), while the y-value is $\frac{2}{3}$ the x-value at the point (6, 4).

- Provide students with a real-world context that represents a linear relationship. Ask them to represent the relationship with an equation or graph to explain whether it is proportional. For example, provide students a situation in which a roofing contractor uses 3 nails for every shingle that she installs. This situation could be represented with the equation $n = 3s$, in which the number of nails, n, is equal to 3 times the number of shingles, s. Graphically, the relationship is represented as follows:



Students should be able to explain that the relationship is proportional because the number of nails is proportional to the number of shingles and the graph of the relationship is in a straight line through the origin.

Key Academic Terms:

input, output, variable, equation, proportional, non-proportional, equation, origin

Additional Resources:

- Activity: Proportional and nonproportional relationships
- Video: Proportional vs. non-proportional (relationships on graphs)

Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
8. Graph proportional relationships. a. Interpret the unit rate of a proportional relationship, describing the constant of proportionality as the slope of the graph which goes through the origin and has the equation $y = mx$ where m is the slope.

Guiding Questions with Connections to Mathematical Practices:

How is the equation for a non-vertical line that passes through the origin derived?

M.P.8. Look for and express regularity in repeated reasoning. Represent the unit rate as m so that when x changes by one unit, y changes by m units. For example, when x changes by 2, y changes by $2m$. Whatever the change in x, the change in y is mx, so $y = mx$. Additionally, demonstrate that a line passing through the origin and a point (x, y) has a unit rate or slope of $\frac{y}{x}$.

- Ask students to derive the equation for a line that passes through the origin and a given point. For example, given that a line passes through the origin and the point $(-4, 5)$, derive the equation $y = -\frac{5}{4}x$ by determining that a straight line through the origin indicates a proportional relationship between x and y. This implies that y-values can be found by multiplying the corresponding x-values by the constant of proportionality. The constant of proportionality for an x-value of -4 and a y-value of 5 is $-\frac{5}{4}$. The equation that indicates that y-values can be found by multiplying the corresponding x-values by $-\frac{5}{4}$ is $y = -\frac{5}{4}x$.

- Ask students to determine whether the line passing through a pair of points also passes through the origin. For example, give students the points $(-2, -3)$ and $(4, 6)$ and ask whether the line passing through both points also passes through the origin. Determine that the slope of the line containing both points is $\frac{3}{2}$. Consider the fact that any line passing through the origin must be of the form $y = mx$, where m is the slope of the line. The points $(-2, -3)$ and $(4, 6)$ are both solutions to $y = \frac{3}{2}x$, so the equation is the correct equation of the line passing through the points given and also passes through the origin.

How is the unit rate of a proportional relationship found from a table, graph, equation, diagram, or verbal description?

M.P.8. Look for and express regularity in repeated reasoning. Use the constant of proportionality to determine the unit rate, which is how much the output value changes when the input value changes by exactly 1 unit. For example, a table shows how many cars are assembled in a factory for different numbers of days. In 2 days, 400 cars are assembled. In 3 days, 600 cars are assembled, and so on. When the input value increases by 1 day (exactly 1 unit), the output value increases by 200. Therefore, the constant of proportionality is 200 cars per day. Additionally, know that the unit rate of a proportional relationship can be determined if the ratio of the change in the dependent variable to the change in the independent variable has a denominator other than 1. This can be found by using an equivalent fraction with a denominator of 1. Further, use the unit rate of a proportional relationship to represent the proportional relationship in a different way.

- Ask students to determine the unit rate of a proportional relationship given as a table, graph, equation, diagram or verbal description. For example, use the following table to determine the constant of proportionality:

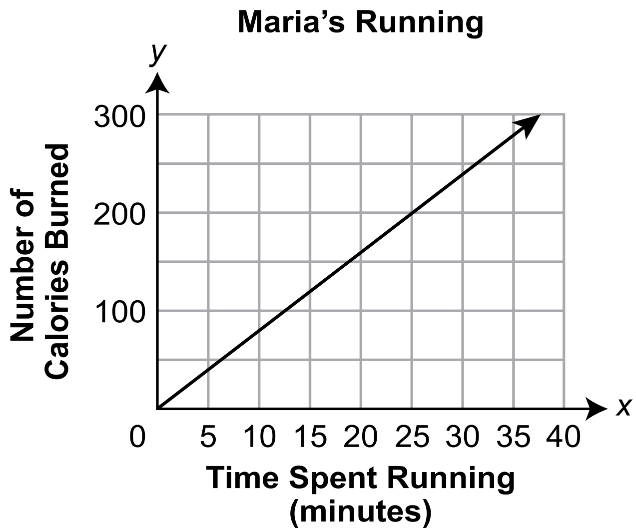
Number of Hamburgers	Cost in Dollars
2	5
4	10
6	15
8	20

I determined that the constant of proportionality/cost per hamburger is \$2.50 by finding that when the number of hamburgers changes by 2, the cost changes by \$5, so when the number of hamburgers changes by 1, the cost changes by \$2.50.

- As an additional example, write an equation based on the given fact that a car averages 25 miles per gallon.

$y = 25x$

- Ask students to use the unit rate of a proportional relationship to represent that relationship in a different way. For example, give students the following graph:



Find how many calories Maria burns per minute while running and use that information to represent the relationship in a different way. The graph of the line includes the origin. Therefore, the ratio of the y-value to the x-value for any point on the line that is not the origin gives the unit rate. Determine that Maria burns 8 calories per minute while running by using a point from the graph such as (25, 200) and using that point to write the ratio $\frac{200}{25} = \frac{8}{1}$. Use this information to represent the relationship as a table, equation, or verbal description. A possible table is shown.

Time (minutes)	Number of Calories
0	0
1	8
3	24
4	32
5	40

How does the slope of a graph connect to the unit rate of a proportional relationship?

M.P.4. Model with mathematics. Represent the unit rate as the slope of the line in a graph. For example, a line that starts at the origin and passes through the point (2, 7) means that as x increases by 2, y increases by 7. Therefore, the slope of the line is $\frac{7}{2}$ and the unit rate is $\frac{7}{2}$. Additionally, demonstrate that the graph of a line that starts at the origin with a slope of m contains all points of the form (n, mn), where n is any positive number. Further, know that the slope of the graph of a line that represents a particular context can be used to represent the unit rate of the given context.

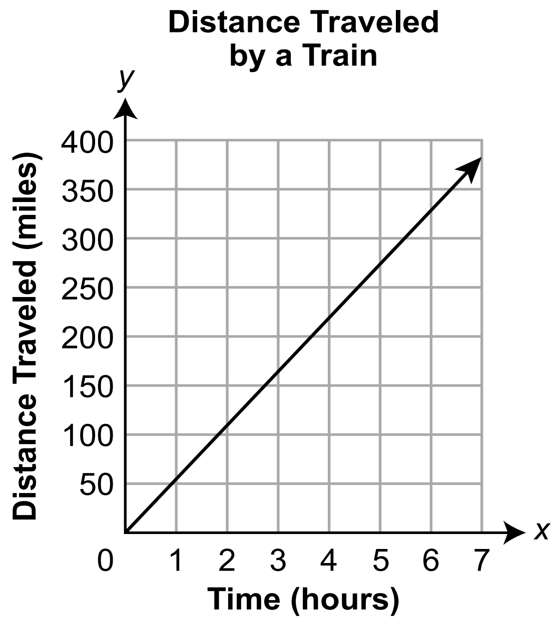
- Ask students to determine the slope of a line that represents a proportional relationship given a point on the line that is not the origin. For example, given that the graph of a proportional relationship passes through the origin and the point (4, 12), determine that the slope of the line is 3. The slope is $\frac{12}{4} = \frac{3}{1} = 3$ because for any line that passes through the origin, the unit rate is the ratio of the y-value to the x-value of any point on the line that is not the origin. The unit rate is the slope.

- Ask students to calculate a set of points that determines a line representing a proportional relationship given the equation of the proportional relationship. For example, given the equation $y = \frac{3}{5}x$, construct a table of points that the graph of the line contains by selecting values of x and multiplying each value by the unit rate of $\frac{3}{5}$.

x	y
5	3
10	6
15	9
20	12

Ask students to confirm that the slope of the line through these points is equal to the unit rate of $\frac{3}{5}$.

- Ask students to determine the unit rate of a given context when provided with a graph representing the context. For example, give students the following graph:



  Determine that the train travels at a unit rate of 55 miles per hour because the slope of the graphed line is 55.

Key Academic Terms:

input, output, variable, equation, proportional, coordinate plane, unit rate, slope, graph, table, constant rate of change, relationship, origin

Additional Resources:

- Activity: Thinkport | Proportional relationships and slope: part 1
- Lesson: Proportional relationships
- Video: Real-life math | Radiologist
- Activity: Gym membership plans
- Lesson: Determining unit rates from graphs
- Lesson: Graphing proportional relationships

Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
9. Interpret $y = mx + b$ as defining a linear equation whose graph is a line with m as the slope and b as the y-intercept. a. Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in a coordinate plane.

Guiding Questions with Connections to Mathematical Practices:

How is the equation $y = mx + b$ connected to the equation $y = mx$ for graphing lines?

M.P.7. Look for and make use of structure. Connect the equation $y = mx$ to $y = mx + b$ by explaining that b represents the y-intercept and, when $b = 0$, the situation represented is proportional. For example, the equation $y = 2x$ graphs a line through the origin that has a slope of 2, and the equation $y = 2x + 3$ graphs a line with the same slope that is shifted 3 units higher and crosses the y-axis at 3, making a line that is parallel to the first line. Additionally, demonstrate that in the equation $y = mx + b$, when $x = 0$, $y = b$, so the ordered pair $(0, b)$ is a point on the graph of $y = mx + b$ and because the x-coordinate is 0, the point represents the y-intercept of the function.

- Ask students to demonstrate how the equations $y = mx$ and $y = mx + b$ are connected for specific values of m and b with $b \neq 0$. For example, give students the equations $y = \frac{5}{2}x$ and $y = \frac{5}{2}x + 3$. Construct a table for both equations, using the same input values for both tables.

$y = \frac{5}{2}x$

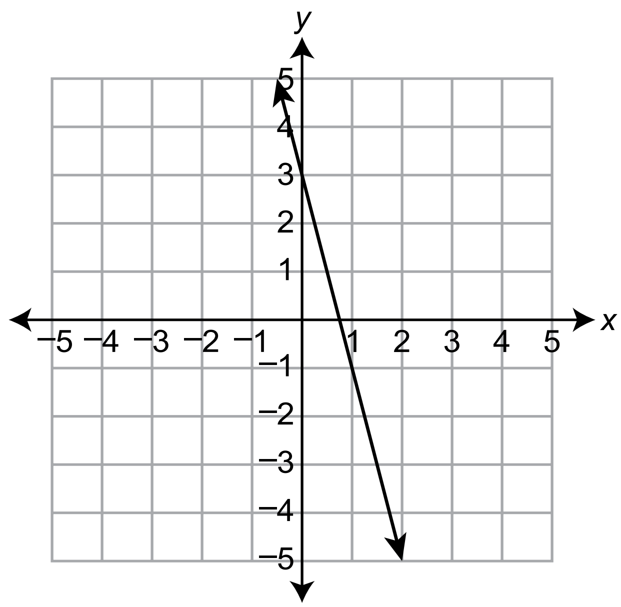
x	**y**
0	0
2	5
4	10
6	15

$y = \frac{5}{2}x + 3$

x	**y**
0	3
2	8
4	13
6	18

Determine that the points in the table from the second equation have y-values that are 3 more than the corresponding y-values from the first equation. The tables have the same rate of change, which means that the graphs have the same slope, but the first equation passes through the y-axis at the origin, while the second equation passes through the y-axis at the point $(0, 3)$. Conclude that an equation of the form $y = mx + b$ has a slope of m and passes through the y-axis at the point $(0, b)$.

- Ask students to write the equation of a line in the form $y = mx + b$ given the graph of the line. For example, give students the graph shown.

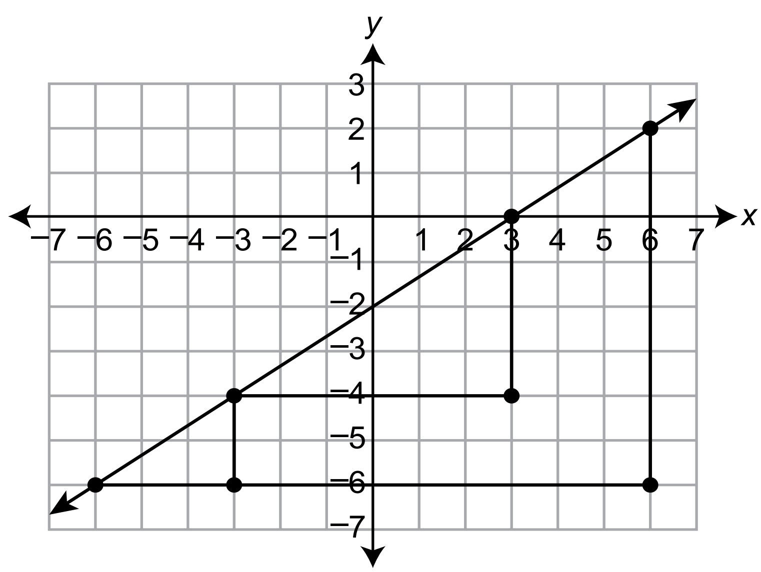


Determine that the equation $y = -4x + 3$ can be used to describe the line by determining that the value of b is 3 because the line passes through the point (0, 3) and the slope of the line, m, is -4 by identifying two points with whole number coordinates on the graph and finding the ratio of the change in the y-values to the change in the x-values.

How can similar triangles show that slope is a constant value on a non-vertical line?

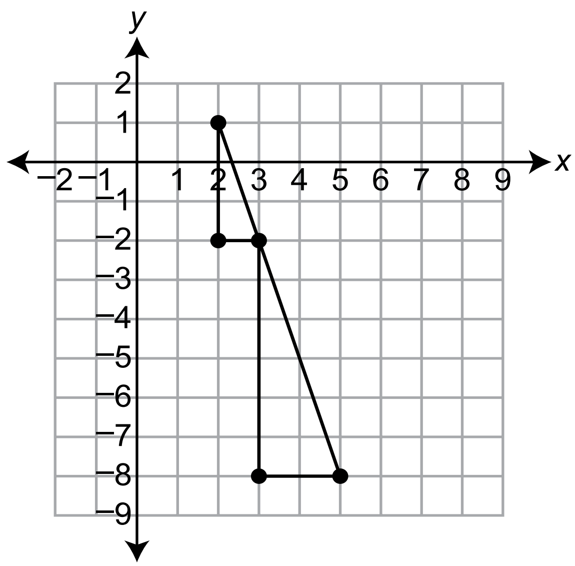
M.P.3. Construct viable arguments and critique the reasoning of others. Represent the slope between any two distinct points on a line by drawing similar right triangles to demonstrate that the slope is the same between any two points. The lengths of the vertical legs represent the change in y-value between two points on the line, and the lengths of the horizontal legs represent the change in x-value between two points on the line. The ratio of the vertical change to the horizontal change is the slope. Because any two of the right triangles are similar, the ratios of the lengths of the vertical and horizontal legs are equivalent; therefore, the slopes are equivalent. For example, a graphed line that goes through the origin and also passes through the points (1, 5), (2, 10), (3, 15), and (4, 20) can have any number of similar right triangles drawn with the line as the hypotenuse, and every right triangle on that line will have the ratio of the change in y over the change in x as a value equivalent to 5. Additionally, know that the converse is also true: using similar right triangles to plot additional points will result in points that all lie along the same line.

- Ask students to show that the slope of a non-vertical line is a constant value using similar triangles. For example, give students the information that a line passes through the points (−6, −6), (−3, −4), (3, 0), and (6, 2). Plot the points and construct similar right triangles using vertical and horizontal line segments between pairs of points as the legs and the line segment between pairs of points as the hypotenuse.



Use the ratio of the length of the vertical leg of each right triangle to the length of the horizontal leg of the corresponding right triangle to show that the slope is a constant value, $\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$.

- Ask students to determine the location of points on the graph of a line given one point on the line and the slope of the line using similar right triangles. For example, given that a line has a slope of −3 and contains the point (2, 1), determine that the line also contains the points (3, −2) and (5, −8) by constructing similar right triangles on the graph. Ask students to describe their steps.



First, I find (3, −2) by drawing the vertical side 3 units down from (2, 1) and then the horizontal side 1 unit right of the given point because the slope can be represented as $\frac{-3}{1}$. Then, I find (5, −8) by drawing the vertical side 6 units down from (3, −2) and the horizontal side 2 units right of that point because $\frac{-6}{2} = \frac{-3}{1}$.

Key Academic Terms:

similar, right triangle, slope, origin, horizontal, vertical, axis, unit rate, graph, input, output, intercept, coordinate plane, *y*-intercept, hypotenuse, slope triangle, ordered pair, point, rate of change

Additional Resources:

- Video: Understanding slope with similar triangles
- Lesson: Applying similar triangles to finding the slope of a linear equation
- Lesson: Applying rates of similar triangles
- Video: Writing the equation of a line when given a graph
- Video: Find those Gleamers | Cyberchase
- Video: Manipulating graphs

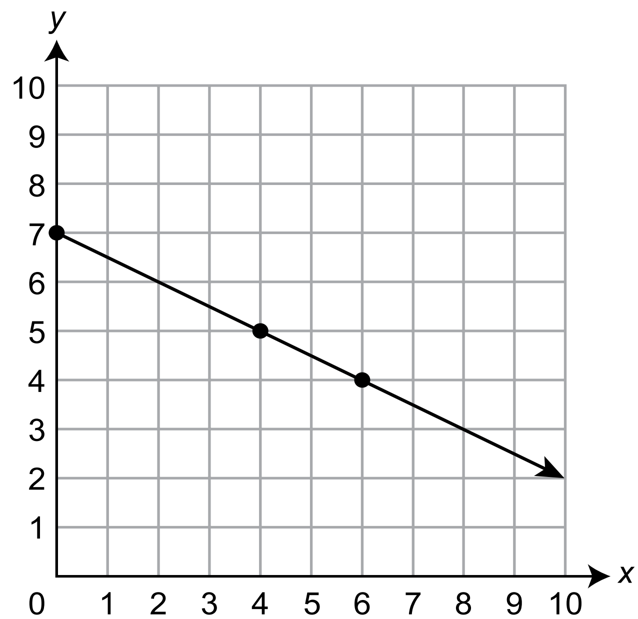
Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
9. Interpret $y = mx + b$ as defining a linear equation whose graph is a line with m as the slope and b as the y-intercept. b. Given two distinct points in a coordinate plane, find the slope of the line containing the two points and explain why it will be the same for any two distinct points on the line.

Guiding Questions with Connections to Mathematical Practices:

Why can any two distinct points on a line be used to the calculate the slope of that line?

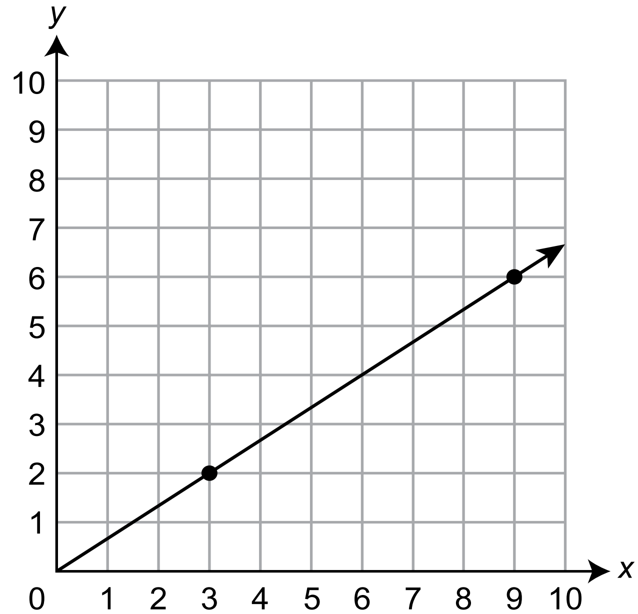
M.P.7. Look for and make use of structure. Observe that when given a straight line on the coordinate plane, the ratio of the vertical change (or "rise") to the horizontal change (or "run") for any two distinct points will be the same. Any two points can be used to determine the slope because any two points on the line can be used to define a right triangle. Any pair of right triangles formed in this way will be similar, and, therefore, the ratios of their legs, which are both the same as the slope, will be the same. For example, the points (2, 1), (4, 2), and (6, 3) are all found on the same line. The difference in y-values (rise) from (2, 1) to (4, 2) is 1, and the difference in x-values (run) is 2. As such, the slope is $\frac{1}{2}$. In the same way, the difference in y-values (rise) from (2, 1) to (6, 3) is 2, and the difference in x-values (run) is 4. As such, the slope is $\frac{2}{4}$, or $\frac{1}{2}$. The ratio between x and y is the same regardless of which two points are used. Additionally, explain why it does not matter which point is designated as (x_1, y_1) and which point is designated as (x_2, y_2) when determining the slope.

- Provide students with three distinct points on the same line. Ask them to determine the slope using the first and second points, the second and third points, and then the first and third points. For example, provide students with a line that passes through the points (0, 7), (4, 5), and (6, 4).



Using the first and second points, the slope can be calculated as $\frac{5-7}{4-0} = \frac{-2}{4} = -\frac{1}{2}$; using the second and third points, the slope can be calculated as $\frac{4-5}{6-4} = \frac{-1}{2} = -\frac{1}{2}$; and using the first and third points, the slope can be calculated as $\frac{4-7}{6-0} = \frac{-3}{6} = -\frac{1}{2}$. Even though the last pair of points are subtracted in reverse, the resulting slope is still $-\frac{1}{2}$. Students should observe that any set of points on the same line can be used in any order to determine the slope of that line.

- Ask students to determine a point on a line when given two distinct points on the same line. For example, provide students with a line that passes through the points (3, 2) and (9, 6), as shown. Ask them to identify a third point (other than the origin) that must be on the same line.



In this case, the slope of the line is $\frac{6-2}{9-3} = \frac{4}{6} = \frac{2}{3}$. A third point, (x_3, y_3), can now be determined using one of the original two points. Based on the point (9, 6), the coordinates of an additional point must be equal to $\frac{6-y_3}{9-x_2} = \frac{2}{3}$. Since $6 - 4 = 2$ and $9 - 6 = 3$, a third point on the line could be (6, 4). Alternatively, students could start at the point (3, 2) and then move up 2 and right 3 to determine the point (6, 4).

Key Academic Terms:

slope, origin, horizontal, vertical, axis, graph, input, output, intercept, coordinate plane, *y*-intercept, rise, run, ratio, ordered pair, point, rate of change, unit rate

Additional Resources:

- Activity: Put the point on the line
- Lesson: Using equations for lines

Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
9. Interpret $y = mx + b$ as defining a linear equation whose graph is a line with m as the slope and b as the y-intercept. c. Graph linear relationships, interpreting the slope as the rate of change of the graph and the y-intercept as the initial value.

Guiding Questions with Connections to Mathematical Practices:

How can the slope and *y*-intercept be identified and interpreted based on an equation that is expressed in slope-intercept form?

M.P.2. Reason abstractly and quantitatively. Know that when a linear relationship is of the form $y = mx + b$, the value of m indicates the slope (i.e., the ratio of the vertical change to the horizontal change between any two points) and the value of b indicates the y-intercept (i.e., the value of y when $x = 0$). For example, in the linear relationship expressed by the equation $y = \frac{1}{2}x - 3$, the $\frac{1}{2}$ indicates a vertical change of 1 for every horizontal change of 2. The −3 indicates that the y-intercept is −3. Additionally, show that if the signs of both the numerator and denominator of the slope are the same, then the slope is positive. If the signs of the numerator and denominator of the slope are different, then the slope is negative.

- Provide students with several equations expressed in slope-intercept form. Ask them to identify the slope and y-intercept of each equation. For example, show students the following equations:

  - $y = -\frac{1}{5}x + 4$
  - $y = \frac{4}{1}x + 5$
  - $y = \frac{4}{5}x - 1$

  In the first equation, the slope is equal to $-\frac{1}{5}$, and the y-intercept is equal to 4. In the second equation, the slope is equal to $\frac{4}{1}$ or 4, and the y-intercept is equal to 5. In the third equation, the slope is equal to $\frac{4}{5}$, and the y-intercept is equal to −1.

- Provide students with several equations expressed in slope-intercept form. Ask them to identify which equations have the same slope and which equations have the same y-intercept. For example, show students the following equations:

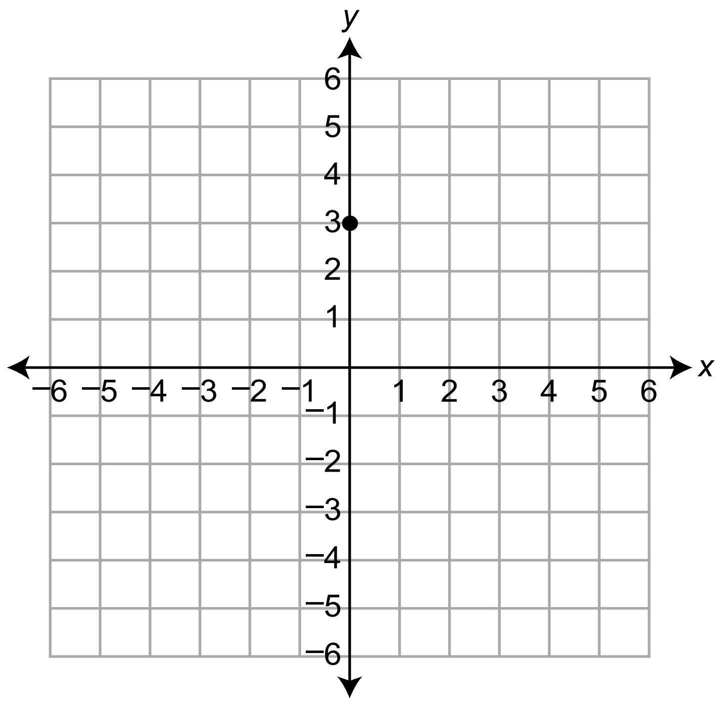
    - $y = \frac{-5}{-4}x + 6$
    - $y = \frac{5}{4}x - 6$
    - $y = \frac{-5}{4}x + 6$
    - $y = \frac{5}{-4}x - 6$

  The first and second equations have the same slope of $\frac{5}{4}$. Likewise, the third and fourth equations have the same slope of $-\frac{5}{4}$. The first and third equations have the same y-intercept of 6. Similarly, the second and fourth equations have the same y-intercept of −6.

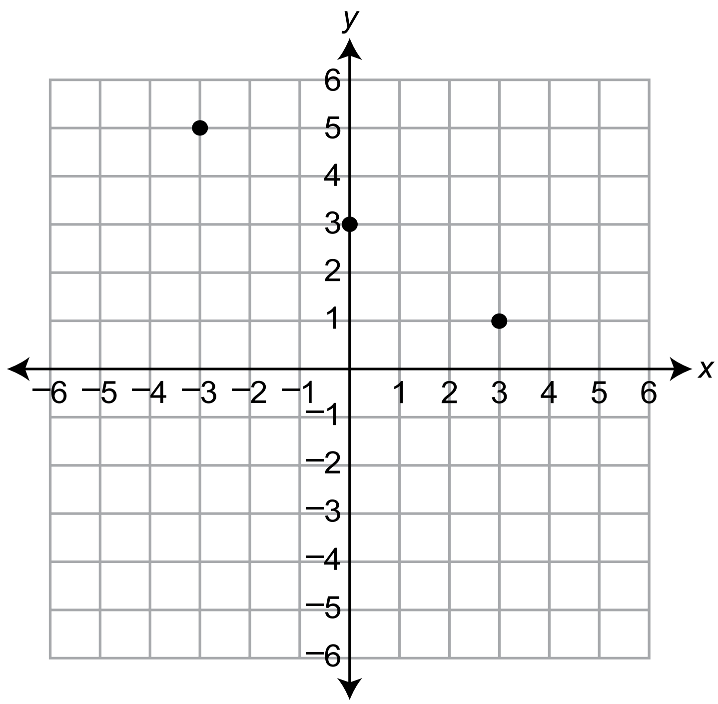
How can a linear relationship expressed in slope-intercept form be graphed without creating a table of values?

M.P.4. Model with mathematics. Know that when given a linear relationship expressed in the form $y = mx + b$, the value of b can first be used to plot the y-intercept at the point $(0, b)$. The value of m can then be used to count up or down (the vertical change, or "rise") and then right or left (the horizontal change, or "run") to plot a second point. This can be repeated to plot additional points. A straight line through the points completes a sketch of the graph. For example, to graph the linear relationship expressed by the equation $y = \frac{2}{3}x + 2$, a point can first be plotted at (0, 2) because the y-intercept is 2. A slope of $\frac{2}{3}$ indicates that the vertical change between any two points on the line is 2 and the horizontal change between those points is 3. Therefore, from the y-intercept of (0, 2), the coordinates for another point on the line can be determined by increasing the y-coordinate by 2 and increasing the x-coordinate by 3. That is, starting from the point (0, 2), students count up 2 (rise) and then right 3 (run) to the coordinate (3, 4). To complete the graph, draw a line through the coordinates (0, 2) and (3, 4). Additionally, demonstrate that when there is no value in the place of b, then the y-intercept is 0 and is located at the origin, (0, 0).

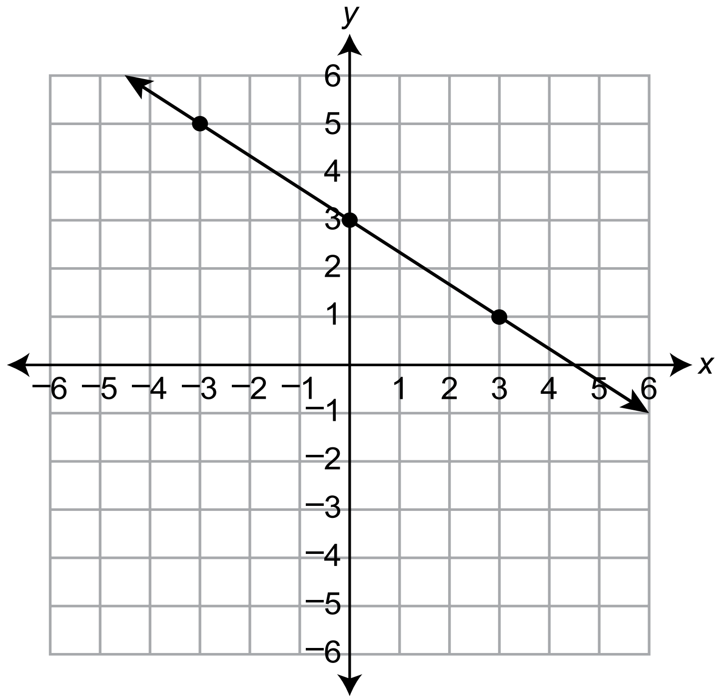
- Ask students to graph a linear relationship expressed in slope-intercept form by identifying and using both the y-intercept and the slope. For example, provide students with the equation $y = -\frac{2}{3}x + 3$. Students can first identify that the y-intercept is 3 and plot a point at (0, 3).



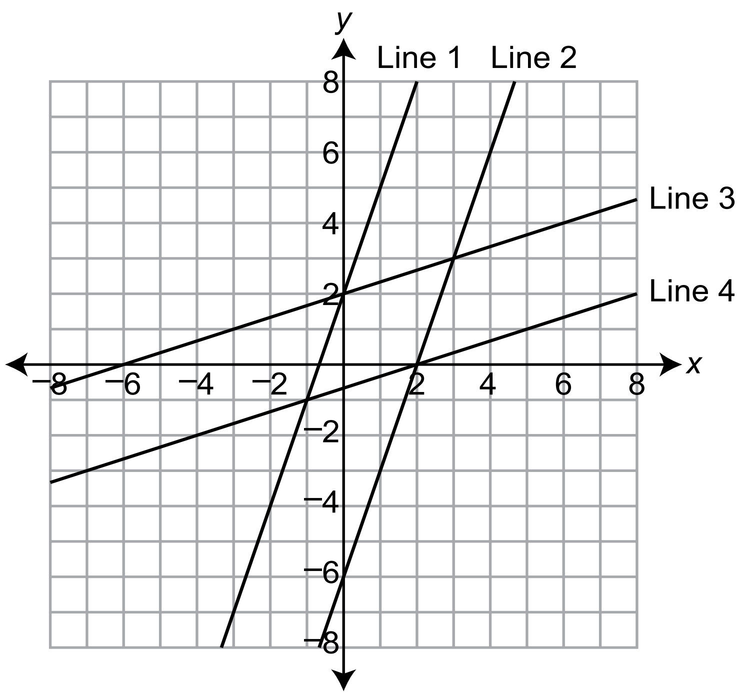
Next, students can identify that the slope is $-\frac{2}{3}$. Since the slope is negative, the numerator and denominator will have opposite signs. Therefore, starting at the point (0, 3), students can decrease the y-coordinate by 2 and increase the x-coordinate by 3 to plot an additional point at (3, 1) or increase the y-coordinate by 2 and decrease the x-coordinate by 3 to plot an additional point at (−3, 5).



The graph can now be completed by drawing a straight line through the points.



- Provide students with a linear relationship that is expressed in slope-intercept form. Ask them to identify the same relationship expressed graphically. For example, provide students with the relationship $y = \frac{1}{3}x + 2$ and the following graphs.



Students should be able to identify that line 3 is the graph that represents $y = \frac{1}{3}x + 2$ because the y-intercept is 2 and the slope is $\frac{1}{3}$.

Key Academic Terms:

slope, origin, horizontal, vertical, axis, graph, input, output, intercept, coordinate plane, y-intercept, linear, rate of change, initial value, rise, run, ordered pair, point, line

Additional Resources:

- Lesson: Translating to $y = mx + b$
- Video: Manipulating graphs

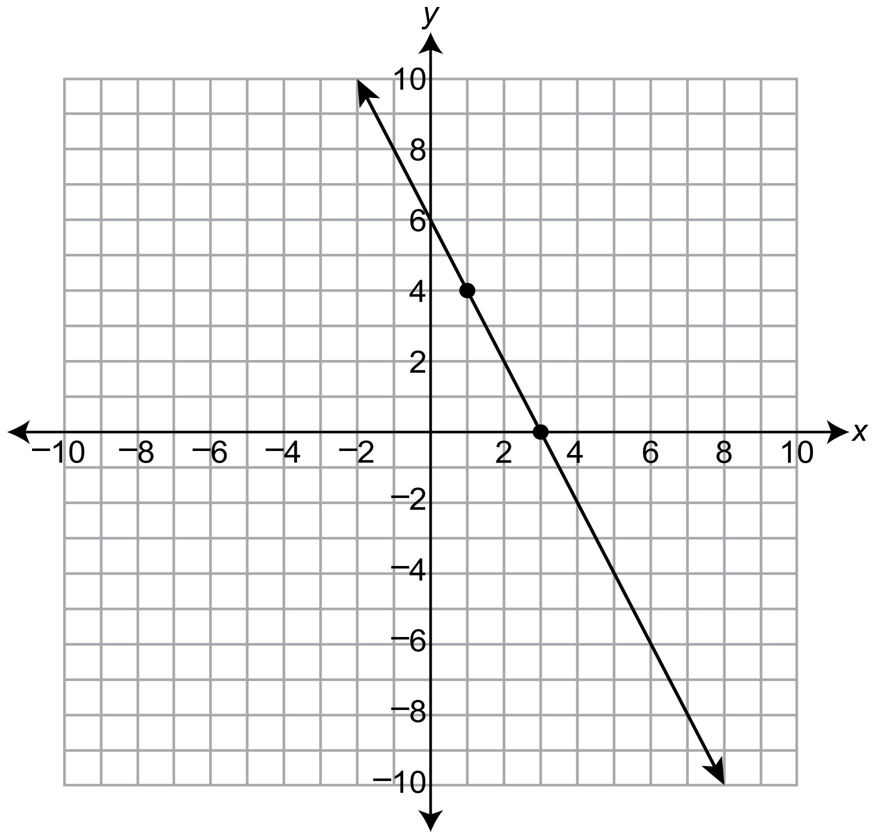
Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
9. Interpret $y = mx + b$ as defining a linear equation whose graph is a line with m as the slope and b as the y-intercept. d. Given that the slopes for two different sets of points are equal, demonstrate that the linear equations that include those two sets of points may have different y-intercepts.

Guiding Questions with Connections to Mathematical Practices:

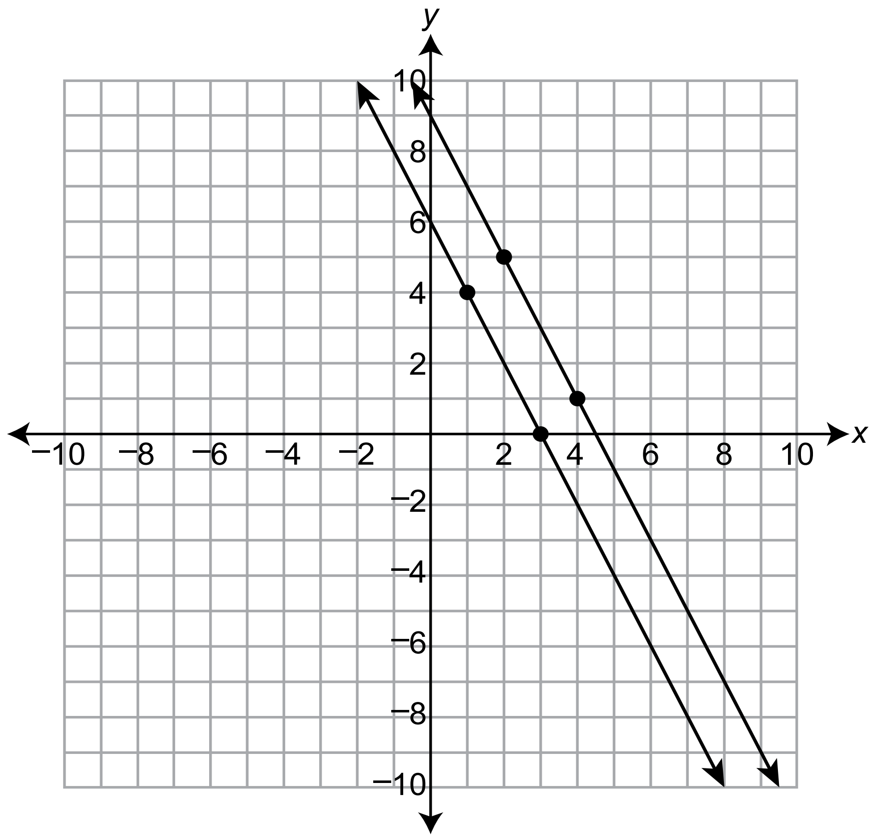
How can a graph be used to determine whether linear equations have the same *y*-intercept if those linear equations pass through two different sets of points and have equal slopes?

M.P.4. Model with mathematics. Know that two different straight lines can be drawn through two different sets of points with equal slopes to determine whether the two sets have the same y-intercept. For example, the set of points (4, 1) and (3, 2) has the same slope as the set of points (1, 2) and (3, 0), which is $^{-}1$. When a straight line is drawn through the first set of points, it can be observed that the y-intercept is 5. When a straight line is drawn though the second set of points, it can be observed that the y-intercept is 3. As such, the corresponding linear equations have the same slope but different y-intercepts. Additionally, observe that if two different sets of points with equivalent slopes have different y-intercepts, then the lines are parallel to one another.

- Provide students with two sets of points that have the same slope. Ask them to demonstrate graphically that the y-intercepts are different. For example, provide students with the set of points (1, 4) and (3, 0) and the set (2, 5) and (4, 1). When a straight line is drawn through the first set of points, the y-intercept is revealed as (0, 6).



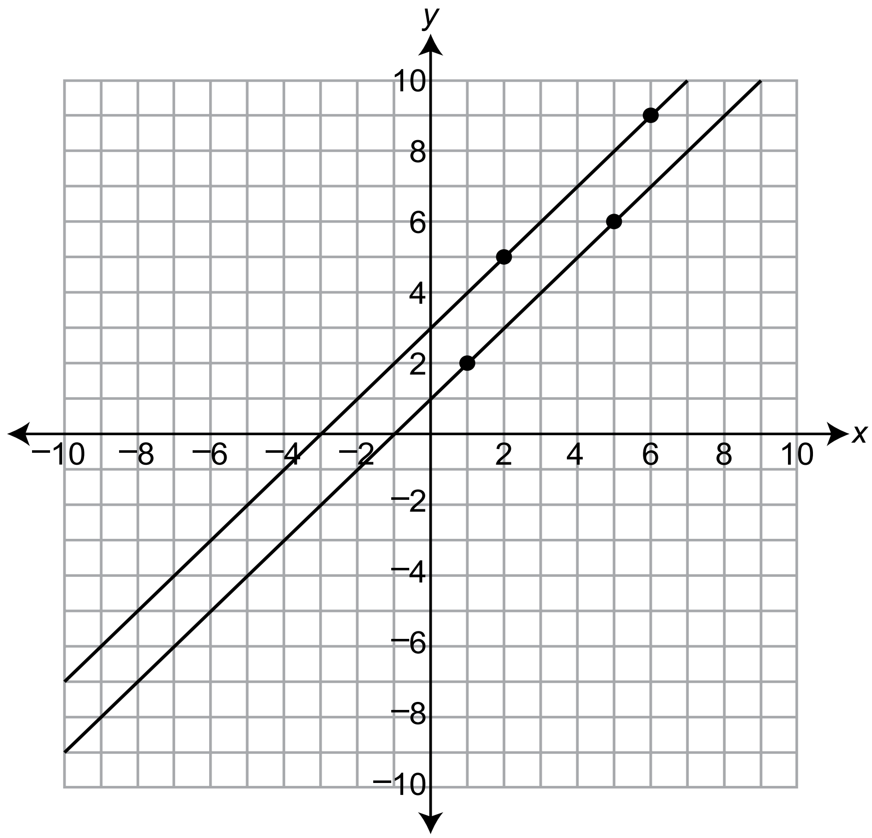
When a straight line is drawn through the second set of points, the *y*-intercept is revealed as (0, 9).



Both sets of points have a slope of −2, but they have different *y*-intercepts.

- Provide students with different sets of points that describe linear relationships. Ask them to determine and explain which sets represent parallel lines. For example, provide students with the following sets of points:
    - Set 1: (1, 2) and (5, 6)
    - Set 2: (2, 5) and (6, 9)

  In this case, the two sets represent parallel lines because the slopes are the same (i.e., 1) and the y-intercepts are different (i.e., 1 and 3). When both lines are graphed, the vertical distance between two points with the same x-coordinate is always 2 units.

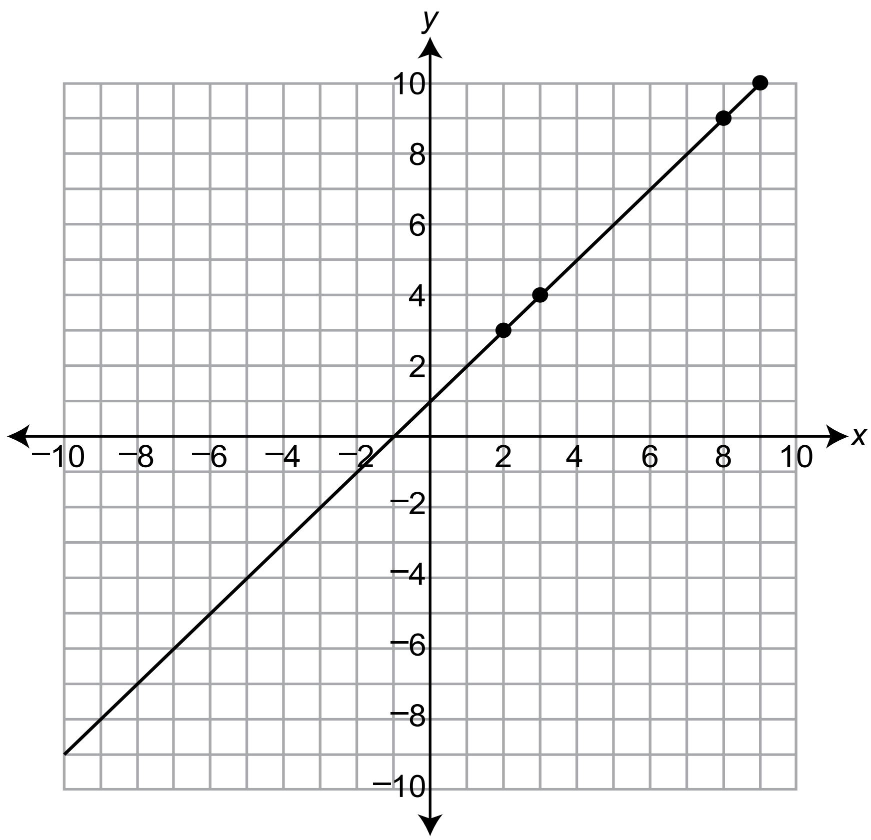


How can algebra be used to determine whether linear equations have the same *y*-intercept if those linear equations pass through two different sets of points and have equal slopes?

M.P.2. Reason abstractly and quantitatively. Know that given two different sets of points with equal slopes, substitution can be used to determine whether the two sets have the same y-intercept. Students substitute one point from each pair and the slope into the equation $y = mx + b$ and then solve for b. For example, the set of points (3, 4) and (9, 6) have the same slope as the set of points (6, 7) and (12, 9), namely $\frac{1}{3}$. When the values $x = 3$, $y = 4$, and $m = \frac{1}{3}$ are substituted into the equation $y = mx + b$ to get $4 = \frac{1}{3}(3) + b$, the value of b is 3. When the values $x = 6$, $y = 7$, and $m = \frac{1}{3}$ are substituted into the equation $y = mx + b$ to get $7 = \frac{1}{3}(6) + b$, the value of b is 5. The y-intercept of the first set of points is 3, which is different from the y-intercept of the second set of points, which is 5. Additionally, observe that if two different sets of points with equal slopes have the same y-intercept, then all the points in both sets are located on the same line.

- Provide students with two different sets of points that represent linear relationships. Ask them to determine, algebraically, whether the slopes and y-intercepts are the same. For example, provide students with the set of points (2, 6) and (4, 2) and the set of points (1, 5) and (3, 1). The slope of each set can be calculated by finding the difference in the y-values divided by the difference in the x-values. The slope of the first set is equal to $\frac{2-6}{4-2}$ or −2. The slope of the second set is equal to $\frac{1-5}{3-1}$ or −2. As such, both sets have the same slope. To determine whether the y-intercepts are the same, one point from each set can be substituted into the equation $y = -2x + b$. For the first set, the point (4, 2) can be substituted into the equation $y = -2x + b$ such that $2 = -2(4) + b$. When the equation is solved for b, it can be determined that the y-intercept is 10. For the second set, the point (3, 1) can be substituted into the equation $y = -2x + b$ such that $1 = -2(3) + b$. When the equation is solved for b, it can be determined that the y-intercept is 7. As such, the two linear relationships have the same slope and different y-intercepts.

- Provide students with two different sets of points that represent linear relationships with the same slope. Ask them to determine whether both sets are located on the same line. For example, provide students with the set of points (2, 3) and (3, 4) and the set of points (8, 9) and (9, 10). Inform students that the slope of the linear relationships represented by both sets is 1. The y-intercept of the first set is located at 1 because regardless of whether (2, 3) or (3, 4) is substituted into $y = 1x + b$, the value of $b = 1$. The y-intercept of the second set is also 1 because regardless of whether (8, 9) or (9, 10) is substituted into $y = 1x + b$, the value of $b = 1$. As such, all four points must be located on the same line. Ask students to verify their conclusion by graphing all 4 points.



Key Academic Terms:

slope, origin, horizontal, vertical, axis, graph, input, output, intercept, coordinate plane, y-intercept, ratio of y to x, rate of change, ordered pair, point, linear, line

Additional Resources:

- Activity: [Parallel lines]
- Activity: [Equation of a line: Dynamic illustrator]

Algebra and Functions
Analyze the relationship between proportional and non-proportional situations.
10. Compare proportional and non-proportional linear relationships represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions) to solve real-world problems.

Guiding Questions with Connections to Mathematical Practices:

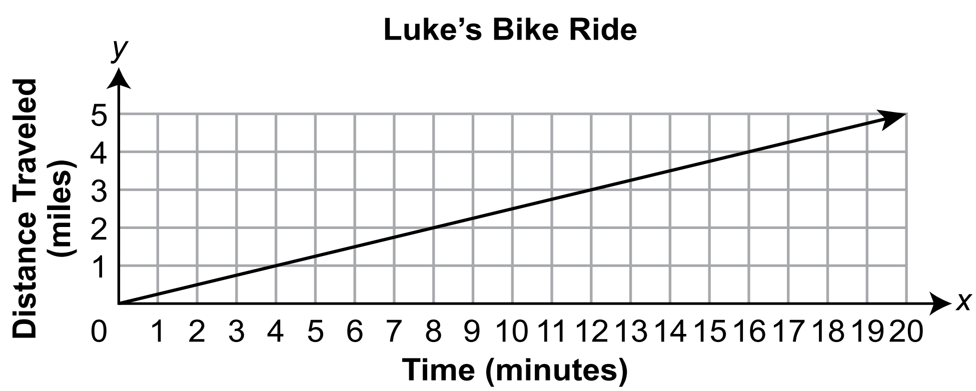
How can two different proportional relationships be compared when represented in different ways?

M.P.6. Attend to precision. Compare proportional relationships represented in different ways by finding the unit rates of both relationships. For example, given one relationship represented by the equation $y = x$ and another relationship represented by a graph of a line with a slope of 3 that goes through the origin, identify that the relationship represented by the graphed line has a rate of change that is 3 times the rate of change of the relationship represented by the equation. Additionally, use the unit rates of proportional relationships represented in different ways to solve problems involving the proportional relationships.

- Give students proportional relationships represented in different ways and ask them to determine which proportional relationship has a greater unit rate and by how much. For example, give students the table and graph shown, both representing the distance traveled by two different bike riders:

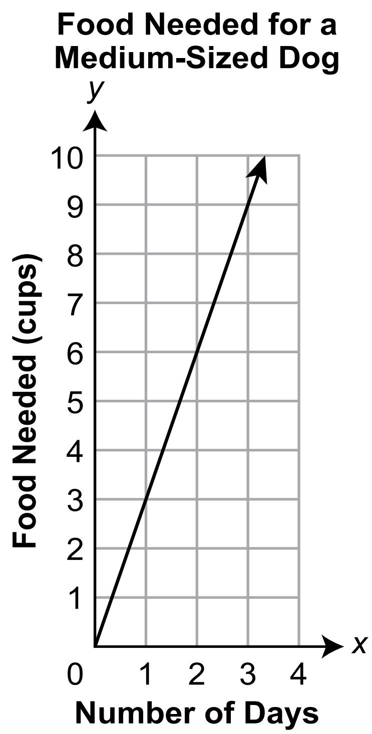
Mario's Bike Ride

Time (minutes)	Distance Traveled (miles)
5	1
10	2
15	3
20	4



Luke's ride, shown on the graph, averages 3 miles per hour faster than Mario's ride, which is shown in the table. Luke completes one mile every 4 minutes of his ride and is therefore traveling 15 miles per hour. Mario completes 1 mile every 5 minutes and is therefore traveling 12 miles per hour.

- Ask students to use the unit rates of proportional relationships given in different ways to solve problems involving those relationships. For example, give students the equation $y = \frac{3}{4}x$, which represents the amount of food in cups, y, a small dog eats over a period of x days. Additionally, give students the graph shown, which represents the amount of food a medium-sized dog eats over a certain number of days.



Ask students to determine how much more food is required to feed a medium-sized dog for 7 days than a small dog for 7 days. A sample student response is shown.

A medium dog needs 15 $\frac{3}{4}$ cups more food than a small dog during a period of seven days. I found the difference in the unit rates by subtracting $\frac{3}{4}$ from 3 and then multiplying the difference by 7.

How can a proportional relationship be compared to a non-proportional relationship?

M.P.4. Model with mathematics. Compare proportional relationships to non-proportional relationships that are represented graphically by identifying the slopes and y-intercepts. For example, the equations $y = 2x$ and $y = 2x + 1$ are both linear and both have a rate of change of $\frac{2}{1}$. However, only the first equation represents a proportional relationship because it passes through the origin and has a constant of proportionality between the x and y values. By contrast, the second equation has a y-intercept located at (0, 1) and does not have a constant of proportionality between the variables. Additionally, solve real-world problems that involve comparing a proportional relationship shown one way (e.g., numerically in a table) to a non-proportional relationship shown another way (e.g., graphically or verbally).

- Ask students to compare a proportional relationship to a related non-proportional relationship. For example, the equation $y = 3x$ represents a proportional relationship, and the equation $y = 3x - 5$ represents a non-proportional relationship. Note that both equations have the same rate of change of $\frac{3}{1}$. That is, the ratio of the change in y-values to the corresponding change in x-values is the same for both relationships for all coordinate pairs. However, the ratio of a y-value to its corresponding x-value, or $\frac{y}{x}$, is not the same for all coordinate pairs in both relationships. The y-value to x-value ratio is only constant in the proportional relationship $y = 3x$ because $\frac{y}{x}$ is always the same value of $\frac{3}{1}$, or 3, for all coordinate pairs (except the undefined case where $x = 0$). The table demonstrates the constant ratios.

$y = 3x$

x	y	$\frac{y}{x}$
−2	−6	3
−1	−3	3
1	3	3
2	6	3

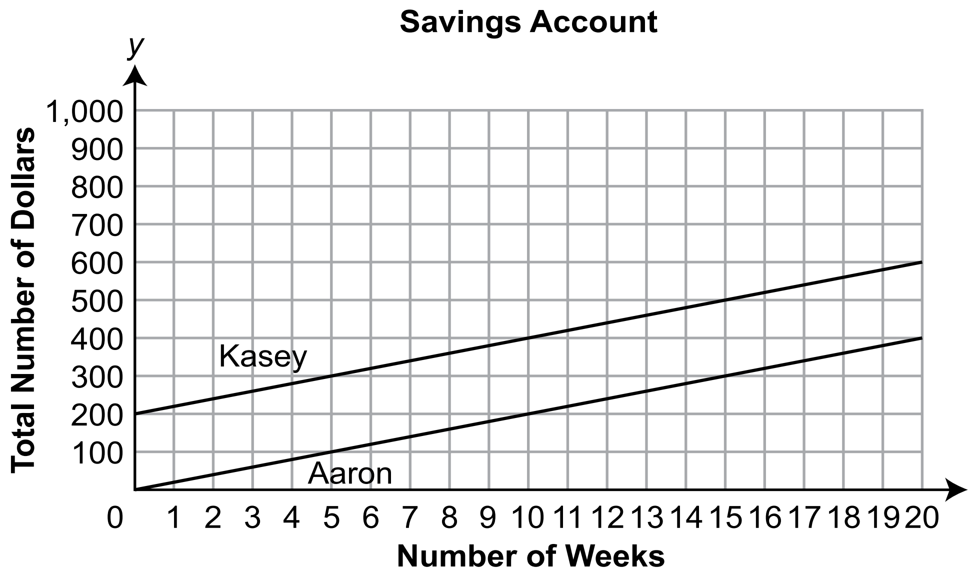
With the non-proportional relationship $y = 3x - 5$, the ratio $\frac{y}{x}$ is not the same for all coordinate pairs. This is demonstrated in the table shown.

$y = 3x - 5$

x	y	$\frac{y}{x}$
−2	−11	$\frac{11}{2}$
−1	−8	8
1	−2	−2
2	1	$\frac{1}{2}$

It can be concluded that proportional relationships have a ratio $\frac{y}{x}$ that is always the same value, called the constant of proportionality, and non-proportional relationships do not.

- Ask students to compare the graphs of a proportional relationship and a non-proportional relationship. For example, the graph shown has two lines, each representing the amount of money in a savings account over time.



The lines are parallel, so they each have a slope of 20. To be a proportional relationship, the graph of the line must pass through the origin. Therefore, the line representing Aaron's savings is proportional, but the line representing Kasey's savings is non-proportional. In the context of the graph, students can see that Kasey's account started with $200 and Aaron started with $0.

Key Academic Terms:

input, output, variable, equation, proportional, non-proportional, coordinate plane, constant of proportionality, unit rate, slope, graph, table, ordered pair

Additional Resources:

- Activity: Thinkport | Proportional relationships and slope: part 1
- Lesson: Proportional relationships
- Video: Real-life math | Radiologist
- Activity: Gym membership plans
- Lesson: Determining unit rates from graphs
- Lesson: Graphing proportional relationships

Algebra and Functions
Analyze and solve linear equations and systems of two linear equations.
11. Solve multi-step linear equations in one variable, including rational number coefficients, and equations that require using the distributive property and combining like terms. a. Determine whether linear equations in one variable have one solution, no solution, or infinitely many solutions of the form $x = a$, $a = a$, or $a = b$ (where a and b are different numbers).

Guiding Questions with Connections to Mathematical Practices:

How are the properties of operations used to solve equations in one variable?

M.P.6. Attend to precision. Use the properties of operations, such as the subtraction property of equality, and the structure of problems to solve linear equations in one variable. For example, $3a + 44 = 194$ can be rewritten as $3a = 150$, which means that $a = 50$ makes the equation true; therefore, 50 is a solution to $3a + 44 = 194$. Additionally, know that equations that are equivalent have the same solution(s).

- Ask students to solve an equation of the form $ax + b = c$, where a, b, and c are rational numbers, and justify the steps used to generate equivalent forms of the equation. For example, give students the equation $\frac{3}{2}x - 4 = 11$. The table shown demonstrates one possible approach. Other approaches are possible.

$\frac{3}{2}x - 4 = 11$	original equation
$\frac{3}{2}x - 4 + 4 = 11 + 4$	addition property of equality
$\frac{3}{2}x + 0 = 15$	additive inverse; add 11 + 4
$\frac{3}{2}x = 15$	additive identity
$\frac{2}{3} \times \frac{3}{2}x = \frac{2}{3} \times 15$	multiplication property of equality
$1x = 10$	multiplicative inverse; multiply $\frac{2}{3} \times 15$
$x = 10$	multiplicative identity

- Ask students to analyze incorrect work used to solve an equation of the form $ax + b = c$, where a, b, and c are rational numbers, determining the correct equivalent equations and correct solution as a result. For example, ask students to identify and correct the errors in the sample work shown.

$$16 = \frac{2}{5}x + 4$$

$$20 = \frac{2}{5}x$$

$$8 = x$$

Determine that when generating the first equation, the student added 4 to the left side of the equation and subtracted 4 from the right side, so the equation that was generated is not equivalent. The correct equivalent equation is $12 = \frac{2}{5}x$. Next, determine that when generating the second equation, the student multiplied the left side by $\frac{2}{5}$ and divided the right side by $\frac{2}{5}$, so the equation that was generated is not equivalent. The correct equivalent equation is $30 = x$. An example approach is shown. Other approaches are possible.

$$\begin{array}{l}
16 = \frac{2}{5}x + 4 \\
\underline{-4 \qquad\quad\;\; - 4} \\
12 = \frac{2}{5}x \\
\underline{\cdot\frac{5}{2} \quad \cdot\frac{5}{2} \qquad} \\
30 = x
\end{array}$$

How can it be determined when an equation with one variable has one solution, infinitely many solutions, or no solutions?

M.P.2. Reason abstractly and quantitatively. Find an equivalent equation in the form of $x = a$, $a = a$, or $a = b$ to determine whether a linear equation has one solution, infinitely many solutions, or no solutions. For example, $t + 2 = 2 + t$ has infinitely many solutions because any value of t yields a true equation, and the equation $h + 3 = h - 4$ has no solutions because any value of h yields a false equation, so there is no value of h for which the equation is true. Additionally, demonstrate that the expressions that make up an equation can be expressed in an equivalent form to determine whether an equation has one solution, infinitely many solutions, or no solutions.

- Ask students to explain why an equation has no solution or infinitely many solutions. For example, give students the equation $b - 3 = {}^-3 + b$. This equation has infinitely many solutions because for any value of b, subtracting 3 or adding $^-3$ yields the same result. Therefore, any value of b used in the equation results in a true statement. In addition, subtracting b from both sides of the equation generates an equivalent equation of $^-3 = {}^-3$. Because $^-3 = {}^-3$ is a statement that is always true, any value can be substituted into the original equation for b to make the equation true. Additionally, give students the equation $5 - w = {}^-w + 8$. This equation has no solutions because for any value of w, subtracting w from 5 and adding ^-w to 8 always yields different numbers. Therefore, there is no value of w that can be used in the equation to make a true statement. In addition, adding w to both sides of the equation generates an equivalent equation of $5 = 8$. Because $5 = 8$ is a statement that is never true, there is no value of w that makes the equation a true statement.

- Ask students to solve equations with infinitely many solutions or no solutions, writing equivalent forms of the expressions that make up the equation if necessary. For example, give students the equation $2(r + 3) - 7 = 5 + 2r - 6$. Determine that the expression $2(r + 3) - 7$ is equivalent to $2r - 1$ and the expression $5 + 2r - 6$ is also equivalent to $2r - 1$.

$$2(r + 3) - 7 \stackrel{?}{=} 5 + 2r - 6$$
$$2r + 6 - 7 \stackrel{?}{=} 5 - 6 + 2r$$
$$2r - 1 \stackrel{?}{=} {}^-1 + 2r$$
$$2r - 1 = 2r - 1$$

infinitely many solutions

Conclude that the equation has infinitely many solutions because the expressions are the same, and therefore any value of r makes the equation a true statement. Additionally, give students the equation $9 - 3(q + 5) = 4q + 8 - 7q$. Determine that the expression $9 - 3(q + 5)$ has an equivalent form of $^-3q - 6$ and the expression $4q + 8 - 7q$ has an equivalent form of $^-3q + 8$. Generate the equivalent equation $^-3q - 6 = {}^-3q + 8$. Conclude that the equation has no solutions because the ^-3q term on each side of the equation will always have the same value, but $^-6$ and 8 will never have the same value, and there are therefore no values of q that make the equation true.

$$9 - 3(q + 5) \stackrel{?}{=} 4q + 8 - 7q$$

$$9 - 3q - 15 \stackrel{?}{=} 4q - 7q + 8$$

$$-3q - 6 \stackrel{?}{=} -3q + 8$$

$$-3q - 6 \neq -3q + 8$$

no solution

How can equations in one variable with rational coefficients be solved?

M.P.6. Attend to precision. Use the properties of operations, the structure of the problem, and/or equivalent expressions to solve for the unknown quantity in an equation. For example, use the properties of operations to solve $\frac{4}{3}c + 7 = \frac{2}{3}c - 1$. First, add the opposites of 7 and $\frac{2}{3}c$ to collect like terms on both sides of the equal sign, resulting in $\frac{4}{3}c - \frac{2}{3}c = {}^-1 - 7$. Combining like terms gives the equivalent equation $\frac{2}{3}c = {}^-8$. Multiplying both sides of the equation by $\frac{3}{2}$ gives the equivalent equation $\frac{3}{2} \bullet \frac{2}{3} \bullet c = \frac{3}{2}\,({}^-8)$. Therefore, $c = {}^-12$. Additionally, observe that the expressions that make up an equation may have equivalent forms that can be generated using the commutative, associative, and distributive properties. Further, observe that an equation maintains its equality when replacing an expression with an equivalent expression or performing the same operation on both sides of the equation.

- Ask students to solve equations containing variable terms on both sides of the equation with rational number coefficients. For example, give students the equation $\frac{2}{3}n + 12 = -18 - \frac{4}{9}n$. There are many ways to solve the equation. One way is to first eliminate the variable term from one side of the equation. Do this by adding $\frac{4}{9}n$ to both sides of the equation to get $\frac{4}{9}n + \frac{2}{3}n + 12 = -18$, and then combine the like terms to generate the equivalent equation $\frac{10}{9}n + 12 = -18$.

$$\begin{array}{rcl} \frac{2}{3}n + 12 & = & -18 - \frac{4}{9}n \\ +\frac{4}{9}n \qquad & & \qquad +\frac{4}{9}n \\ \hline \frac{10}{9}n + 12 & = & -18 \end{array}$$

Next, begin to isolate the variable by subtracting 12 from both sides of the equation to generate the equivalent equation $\frac{10}{9}n = -30$. Lastly, multiply both sides by $\frac{9}{10}$ because the product of $\frac{10}{9}$ and $\frac{9}{10}$ is the multiplicative identity value 1, which yields the solution $n = -27$.

$$\begin{array}{rcl} \frac{10}{9}n + 12 & = & -18 \\ -12 & & -12 \\ \hline \frac{10}{9}n \qquad & = & -30 \\ \cdot \frac{9}{10} \qquad & & \cdot \frac{9}{10} \\ \hline n & = & -27 \end{array}$$

Verify this solution by substituting the value −27 in place of n in the original equation $\frac{2}{3}(-27) + 12 = -18 - \frac{4}{9}(-27)$. Then, use the order of operations to rewrite the expressions on each side of the equal sign.

$$\begin{array}{c} \frac{2}{3}(-27) + 12 = -18 - \frac{4}{9}(-27) \\ -18 + 12 = -18 + 12 \\ -6 = -6 \end{array}$$

This demonstrates that $n = -27$ does make the equation true and is the correct solution.

- Ask students to solve equations containing variable terms on both sides of the equal sign when one or both expressions that make up the equation have equivalent forms that can be generated using the distributive property. For example, give students the equation $5 - 2(2y - 4) = {}^{-}3y - 5 + 5y$. Generate an equivalent expression for $5 - 2(2y - 4)$ by using the distributive property and combining like terms to get $13 - 4y$.

$$5 - 2(2y - 4)$$

$$5 - 4y + 8$$

$$5 + 8 - 4y$$

$$13 - 4y$$

Generate an equivalent expression for ${}^{-}3y - 5 + 5y$ by combining the like terms to get $2y - 5$.

$$^{-}3y - 5 + 5y$$

$$^{-}3y + 5y - 5$$

$$2y - 5$$

Use the equivalent expressions generated to write the equivalent equation $13 - 4y = 2y - 5$. Add $4y$ to both sides of the equation to eliminate the variable term from one side of the equation and get $13 = 6y - 5$. Then, add 5 to both sides of the equation and divide both sides of the equation by 6 to conclude that the solution is $3 = y$.

$$\begin{array}{rcl} 13 - 4y & = & 2y - 5 \\ +\,4y & & +\,4y \\ \hline 13 & = & 6y - 5 \\ +\,5 & & +\,5 \\ \hline \frac{18}{6} & = & \frac{6y}{6} \\ 3 & = & y \end{array}$$

Verify this solution by substituting the value of 3 in place of y in the original equation and concluding that because the resulting equation is a true statement, the solution is correct.

$$5 - 2(2(3) - 4) = {}^{-}3(3) - 5 + 5(3)$$

$$5 - 2(6 - 4) = {}^{-}9 - 5 + 15$$

$$5 - 2(2) = {}^{-}14 + 15$$

$$5 - 4 = 1$$

$$1 = 1$$

Key Academic Terms:

equation, variable, linear, equivalent, constant, solution, infinite, simultaneous linear equations, properties of equality, like terms, distributive property, coefficient, reciprocal, unknown quantity, expressions, properties of equality

Additional Resources:

- Video: Solving equations—with algebra tiles
- Lesson: Algebra tiles and equation solving
- Activity: Solving equations
- Activity: Coupon versus discount
- Lesson: Solving linear equations in one variable using integers
- Lesson: Inverses as a form of undoing
- Article: Solving equations in middle school math
- Video: Thinkport | Solving linear equations
- Lesson: A concrete introduction to the abstract concepts of integers and algebra using algebra tiles

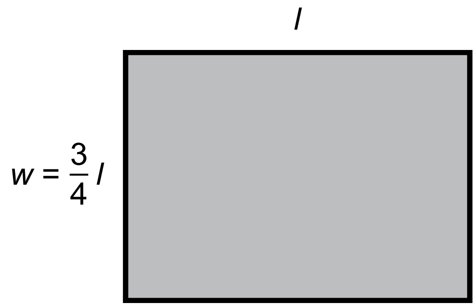
Algebra and Functions
Analyze and solve linear equations and systems of two linear equations.
11. Solve multi-step linear equations in one variable, including rational number coefficients, and equations that require using the distributive property and combining like terms. b. Represent and solve real-world and mathematical problems with equations and interpret each solution in the context of the problem.

Guiding Questions with Connections to Mathematical Practices:

How can real-world problems be represented and solved using linear equations?

M.P.1 Make sense of problems and persevere in solving them. Represent an unknown quantity within a context using a variable and then indicate how it relates to known quantities using operators. For example, give students a scenario in which a teacher ordered some calculators priced at $16.50 per calculator. The total cost of the order, including a flat fee of $8.00 for shipping, was $206. In this context, the number of calculators is unknown and can be represented by the variable c. The total cost of $206 is equal to the cost of purchasing the calculators (the number of calculators times the cost per calculator of $16.50) plus the shipping fee of $8.00. The corresponding equation is $206 = 16.50c + 8$. This equation can then be solved to determine that the value of c is 12 and interpreted to mean that c represents the number of calculators purchased when the total cost is $206. Additionally, identify when the solution to an equation does not make sense contextually.

- Provide students with information about the perimeter of a rectangular object and the relationship between the length and the width of the object. Ask them to create a corresponding equation and solve it to determine the object's dimensions. For example, ask students to create an equation that could be used to find the length and width of a rectangular area rug that has a perimeter of 28 feet and a width that is $\frac{3}{4}$ of the rectangle's length.



  In this case, the perimeter of 28 feet is equal to $l + l + w + w$. Since the width is $\frac{3}{4}$ of the length, the context can be represented with the equation $28 = l + l + \frac{3}{4}l + \frac{3}{4}l$. The equation can be rewritten as $28 = 2l + \frac{3}{2}l$ and rewritten again as $28 = \frac{7}{2}l$. When both sides are multiplied by $\frac{2}{7}$, the result is $8 = l$, or a length of 8 feet. The width must be 6 feet because $\frac{3}{4} \times 8 = 6$. Therefore, the dimensions of the rectangular area rug are 8 feet by 6 feet.

- Ask students to create, solve, and interpret a multi-step equation that represents a monetary transaction involving dollars and cents. For example, a customer has $24 to spend on a bouquet at a florist that charges $9.75 for a vase plus $1.25 per flower. Ask students to create an equation and then solve it to determine how many flowers, f, the customer can buy in a bouquet. Students can create the equation $24 = 1.25f + 9.75$. When solved, the value of f is 11.4. In this context, the customer can only buy 11 flowers because it is not possible to buy 0.4 of a flower.

Key Academic Terms:

equation, variable, linear, equivalent, constant, solution, infinite, simultaneous linear equations, properties of equality, like terms, distributive property, coefficient, reciprocal, unknown quantity, expressions

Additional Resources:

- Video: Solving equations—with algebra tiles
- Lesson: Algebra tiles and equation solving
- Activity: Solving equations
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- Lesson: Solving linear equations in one variable using integers
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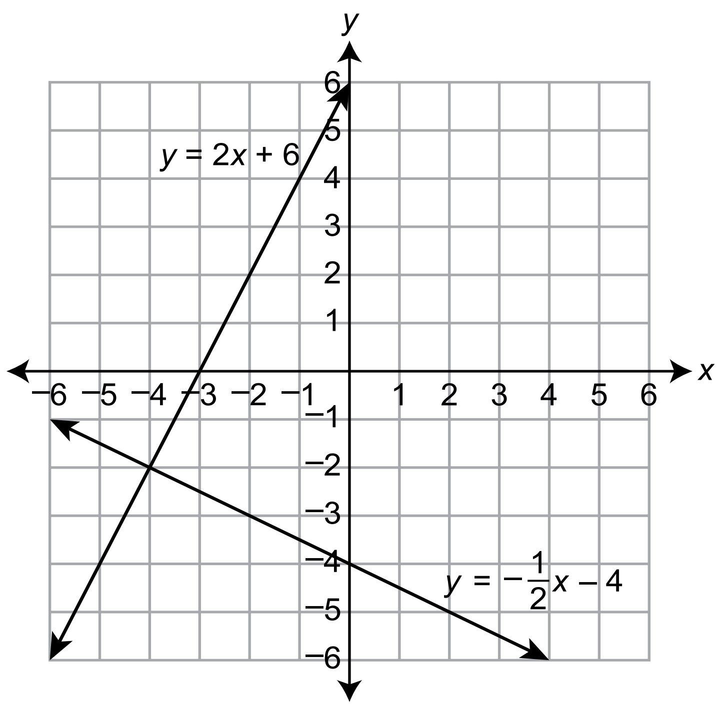
Algebra and Functions
Analyze and solve linear equations and systems of two linear equations.
12. Solve systems of two linear equations in two variables by graphing and substitution. a. Explain that the solution(s) of systems of two linear equations in two variables corresponds to points of intersection on their graphs because points of intersection satisfy both equations simultaneously.

Guiding Questions with Connections to Mathematical Practices:

How does the point where two lines intersect on a graph help solve a system of equations?

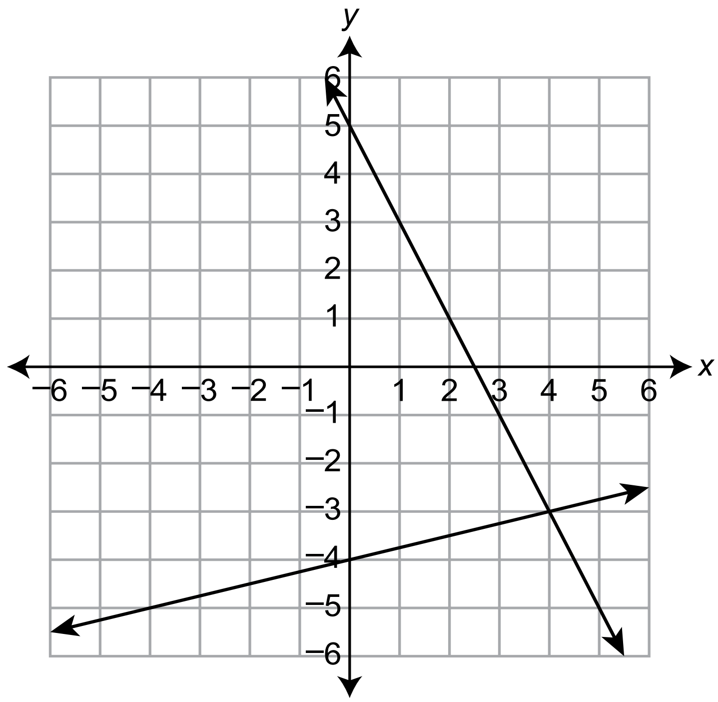
M.P.4. Model with mathematics. Observe that the solution to a system of equations is the intersection of the lines of the two equations at the coordinates (x, y). For example, the graphs of $y = 3x$ and $y = 5x - 2$ intersect at the point (1, 3). The solution to the system of equations is $x = 1$ and $y = 3$ because those values are solutions to both equations, as $3 = 3(1)$ and $3 = 5(1) - 2$. Additionally, observe that for a linear system of equations, the sole intersection point of the graphed lines is the only solution to both equations when the lines are not parallel or coinciding.

- Ask students to solve a system of linear equations by graphing the lines of the two equations, identifying the intersection point, and verifying the solution algebraically. For example, give students the system of equations consisting of $y = 2x + 6$ and $y = -\frac{1}{2}x - 4$. Graph the lines for each equation by plotting the y-intercept, using the slope to identify more points to graph, and connecting those points.



Determine that the solution to the system of equations is (−4, −2) because the lines intersect at (−4, −2). Verify the solution by substituting −4 into both equations for x and −2 into both equations for y. Show that the solution is correct because $-2 = 2(-4) + 6$ and $-2 = -\frac{1}{2}(-4) - 4$.

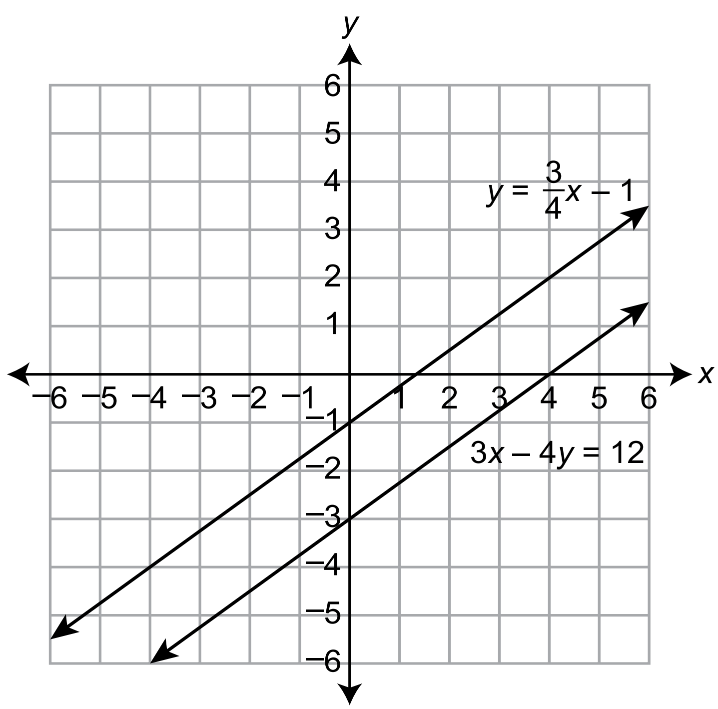
- Ask students to determine the system of equations represented by a given graph and find the solution to that system. For example, give students the following graph.



Determine that the system of equations represented is given by the equations $y = -2x + 5$ and $y = \frac{1}{4}x - 4$ because the first line passes through the y-axis at 5 and has a slope of −2 and the second line passes through the y-axis at −4 and has a slope of $\frac{1}{4}$. The lines intersect at (4, −3), so the solution to the system is (4, −3).

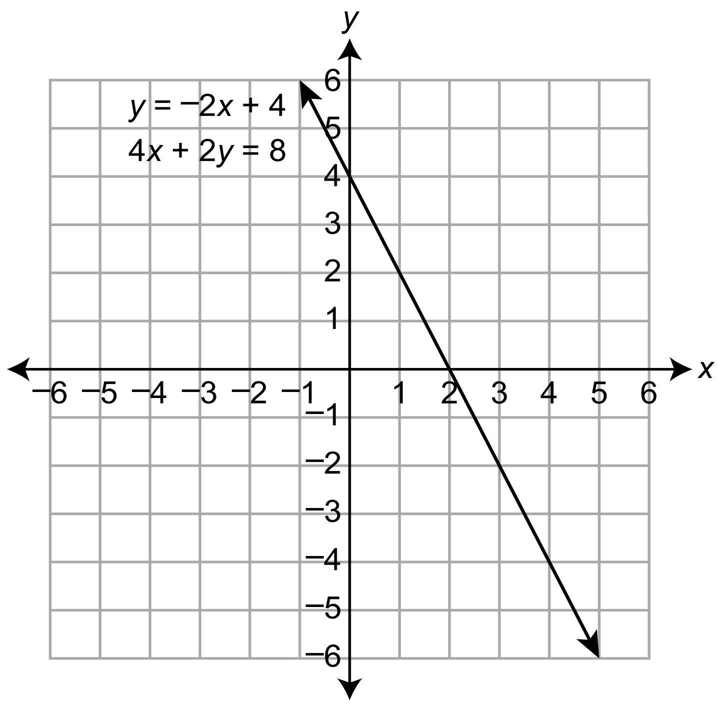
M.P.4. Model with mathematics. Observe that when two equations have the same line, the solution to the system of equations is every point on the line. For example, the graphs of $y = 1.5x + 8$ and $y = \frac{3}{2}x + 8$ intersect at every value of x and y, so every point on the graphs is a solution to both equations. Additionally, observe that when two equations are represented by lines that are parallel, the system of linear equations has no solution.

- Ask students to solve systems of equations with no solution or infinitely many solutions by graphing the lines for both equations. For example, give students the equations $y = \frac{3}{4}x - 1$ and $3x - 4y = 12$. Graph $y = \frac{3}{4}x - 1$ by plotting the y-intercept at $(0, -1)$, using the ratio given by the slope to find the point $(4, 2)$, which is 3 units above and 4 units to the right of the y-intercept, and connecting the points to graph the line. Graph the line given by $3x - 4y = 12$ by identifying the x- and y-intercept as $(4, 0)$ and $(0, -3)$ respectively and then connecting the points to graph the line.



Determine that the system of equations has no solution because the lines are parallel and therefore do not intersect.

As an additional example, give students the equations $y = -2x + 4$ and $4x + 2y = 8$. Graph $y = -2x + 4$ by plotting the y-intercept at (0, 4), using the ratio given by the slope to find the point (1, 2), which is 2 units below and 1 unit to the right of the y-intercept, and connecting the points to graph the line. Graph the line given by $4x + 2y = 8$ by identifying the x- and y-intercept as (2, 0) and (0, 4) respectively and then connecting the points to graph the line.



Determine that the lines coincide and therefore every point on one line is also on the other line. Conclude that the system of equations has an infinite number of solutions, where each solution is a point on the coinciding lines.

- Ask students to determine the number of solutions that a system of equations has by rewriting the equations in the form $y = mx + b$. When the equations in this form are identical, the system has an infinite number of solutions because the graphs of the equations are coinciding lines; when the equations have the same value of m, but a different value of b, the system has no solutions because the graphs of the equations are parallel lines that never intersect; and when the equations in this form have different values of m, the system has exactly one solution because the graphs of the equations form lines that intersect at one point. For example, give students the equations $y = -x + 3$ and $x + y = 2$. Generate an equivalent equation for $x + y = 2$ by subtracting x from both sides, yielding $y = -x + 2$. Determine that the system of equations has no solutions because the slope of both equations is -1 and the equations have different y-intercepts, meaning that the lines representing the equations are parallel. The system has no solutions because parallel lines have no intersection points and there are therefore no points that are solutions to both equations.

Key Academic Terms:

linear equation, intersection, graph, variable, system of equations, solution, coordinate, properties of equality, rate of change, y-intercept, ordered pair

Additional Resources:

- Activity: Fixing the furnace
- Activity: Summer swimming
- Activity: The intersection of two lines
- Video: Graphing a system of equations

Algebra and Functions
Analyze and solve linear equations and systems of two linear equations.
12. Solve systems of two linear equations in two variables by graphing and substitution. b. Interpret and justify the results of systems of two linear equations in two variables (one solution, no solution, or infinitely many solutions) when applied to real-world and mathematical problems.

Guiding Questions with Connections to Mathematical Practices:

How can the structure of the equations in a system of equations be used to determine a strategy for finding a solution?

M.P.1. Make sense of problems and persevere in solving them. Analyze the structure of the given linear equations to plan a solution pathway. For example, the structure of the equations $y = 5 - 3x$ and $2x + 2y = 7$ makes substitution a clear method for solving the system because one of the equations is already solved for one of the variables. Additionally, know that an equivalent form for one or both of the equations that make up a system of equations can be generated so that a particular solution method can be used.

- Ask students to solve a system of equations algebraically using the substitution method. For example, give students the system of equations $y = -4x + 1$ and $2x - 3y = 11$. Determine that the substitution method is a good candidate because the first equation is solved for a variable. Substitute the expression $-4x + 1$ into the second equation for y because the first equation states that y and $-4x + 1$ are equivalent.

  $$2x - 3(-4x + 1) = 11$$

  Generate an equivalent expression for the left side of the resulting equation by first distributing -3 and then combining the like terms, $2x$ and $12x$.

  $$\begin{aligned} 2x + 12x - 3 &= 11 \\ 14x - 3 &= 11 \end{aligned}$$

Solve the equation for x by adding 3 to both sides of the equation and then dividing both sides of the equation by 14.

$$\begin{array}{r} 14x - 3 = 11 \\ +3 \quad +3 \\ \hline \end{array}$$

$$\frac{14x}{14} = \frac{14}{14}$$

$$x = 1$$

Substitute the x-value into the first equation in order to solve for the y-value of the solution to the system of equations. The first equation is used because it is solved for y.

$$y = -4(1) + 1$$

Calculate the value of the right side of the equation to get the y-value of the solution.

$$y = -4 + 1$$

$$y = -3$$

Write the solution as $(1, -3)$.

- Ask students to solve a system of equations that represent parallel or coinciding lines. For example, give students the system of equations $x - 2y = 4$ and $-x + 2y = -3$. Students determine that the first equation can be solved for x and written as $x = 4 + 2y$. Students substitute this new expression for x into the second equation, as shown.

$$-x + 2y = -3$$

$$-(4 + 2y) + 2y = -3$$

$$-4 - 2y + 2y = -3$$

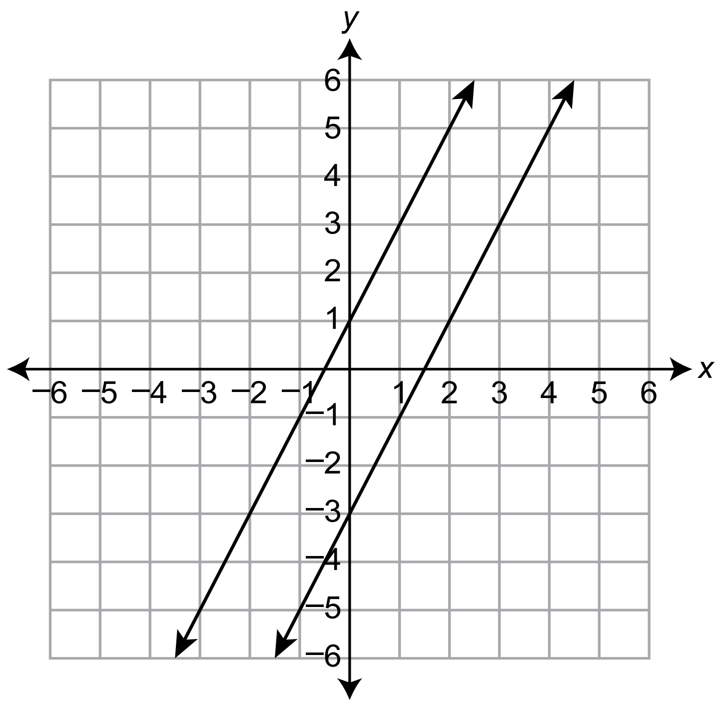
$$-4 = -3$$

This yields the equation $-4 = -3$, which cannot be true. Therefore, the system has no solutions and there are no ordered pairs that are solutions to both equations. From this the conclusion can be made that the equations represent parallel lines on a graph.

How do the graphs of two linear equations help estimate the solution?

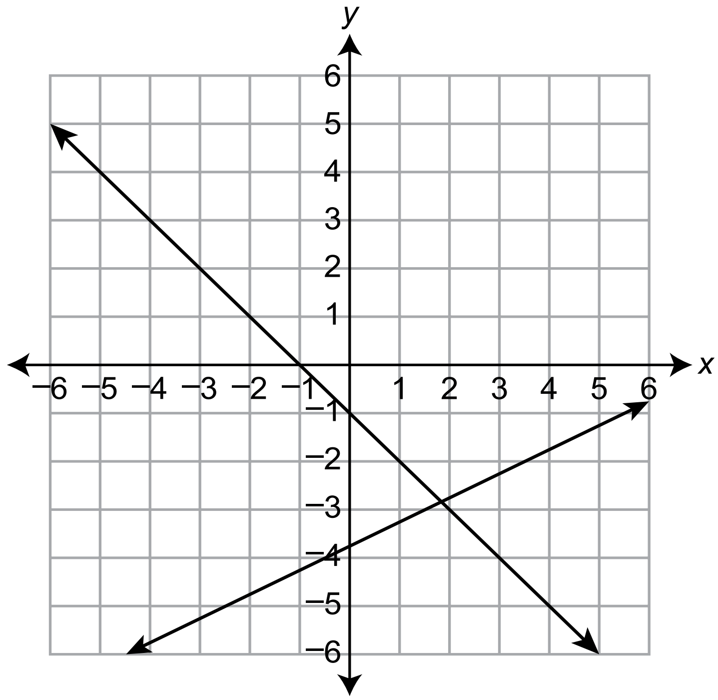
M.P.4. Model with mathematics. Use a graph to estimate solutions to systems of linear equations. For example, a graph with two parallel lines has no solution. Additionally, be aware that a solution that has been estimated using a graph of a system of equations may not give exactly equal values when substituted into one or both equations.

- Ask students to estimate the solution to a system of linear equations using the graph of the system. For example, give students the graph shown.



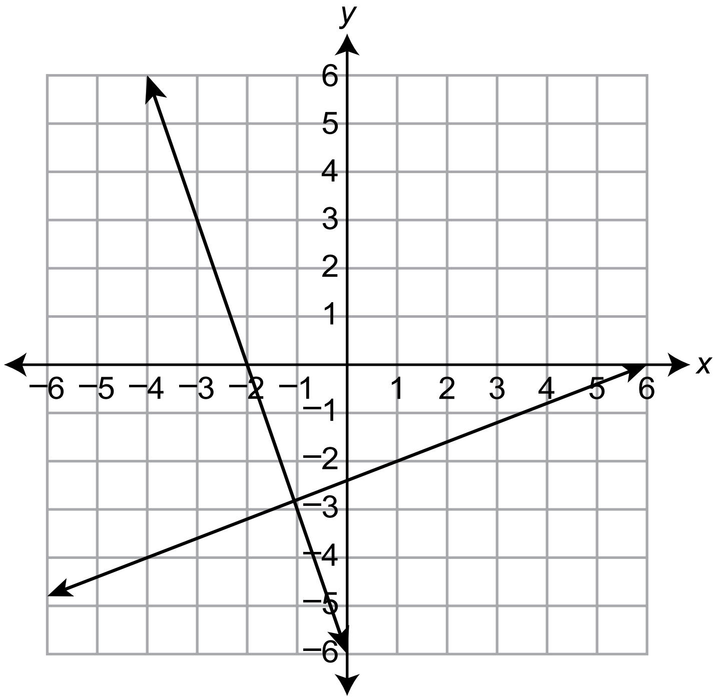
Determine that the system of equations has no solution because the lines appear to be parallel.

Additionally, give students the following graph.



Determine that a reasonable estimate for the solution to the system of equations is (2, −3).

- Ask students to determine whether an estimated solution to a system of linear equations is reasonable by substituting the solution into both equations. For example, give students the graph of the system of equations given by $2x - 5y = 12$ and $-3x - y = 6$.



Estimate the solution to be (−1, −3). Determine whether the estimate is reasonable by substituting the values into both equations for x and y. The relatively close values on both sides of the first equation and the equal values on both sides of the second equation mean it is a reasonable estimate.

$$\begin{aligned} 2(-1) - 5(-3) &= 12 \\ -2 + 15 &= 12 \\ \hline 13 &= 12 \end{aligned} \qquad \begin{aligned} -3(-1) - (-3) &= 6 \\ 3 + 3 &= 6 \\ \hline 6 &= 6 \end{aligned}$$

How do systems of linear equations help solve real-world mathematical problems?

M.P.4. Model with mathematics. Represent and solve situations with two linear equations in two variables. For example, represent the situation "Fred is four times as old as his sister. In 12 years, he will be 4 less than twice her age." with the equation $4s = f$ for the present day and $2(s + 12) - 4 = f + 12$ for 12 years from now. This system of equations can be solved algebraically by substituting the $4s$ for f in the equation that represents 12 years from now. With the substitution, the equation becomes $2(s + 12) - 4 = 4s + 12$. Using the distributive property, the next step to solving is $2s + 24 - 4 = 4s + 12$, and then combining like terms gives $2s + 20 = 4s + 12$. The subtraction property of equality is next, which gives $2s = 8$, and then the division property of equality gives $s = 4$. In the context of this situation, s is the age of Fred's sister in the present day, so that means his sister is 4 years old. Use $s = 4$ in the equation $4s = f$ to find Fred's age: $4 \bullet 4 = f$, or $16 = f$. Fred is 16 years old. Additionally, use key words and phrases to model a given situation with equations. Further, identify which quantities in a situation are represented by each variable in a system of linear equations.

- Ask students to write a system of linear equations that can be used to represent a given situation. For example, give students the situation "One batch of muffins uses 3 eggs and 2 cups of flour. One batch of rolls uses 1 egg and 2 cups of flour. Anita used 11 eggs and 10 cups of flour with no ingredients left over. How many batches of each item did Anita make?" Determine that variables are needed to represent the number of batches of muffins, m, and the number of batches of rolls, r. Write one equation using the amount of eggs Anita used, $3m + 1r = 11$, and write a second equation using the amount of flour that Anita used, $2m + 2r = 10$.

- Ask students to solve real-world problems using a system of linear equations. For example, give students the situation "Mark and Andrew are running in a race. Mark has 200 meters left until he reaches the finish line and Andrew has 175 meters left until he reaches the finish line. Mark is running at a speed of 8 meters per second and Andrew is running at a speed of 6 meters per second. Determine whether Mark will catch up to Andrew. If Mark will catch up to Andrew, determine how many seconds it will take and the distance from the finish line both people will be at that time." Determine that the equation $y = 200 - 8x$ can be used to represent Mark's distance, y, to the finish line after x seconds and $y = 175 - 6x$ can be used to determine Andrew's distance, y, to the finish line after x seconds. Use the substitution method to solve the system because at least one of the equations contains a variable that has been isolated. Substitute the expression $200 - 8x$ into the second equation for y, as the first equation states that $200 - 8x$ and y are equivalent.

  $$200 - 8x = 175 - 6x$$

  Solve the equation for x by adding $8x$ to both sides, then subtracting 175 from both sides, and finally dividing both sides by 2 to get $12.5 = x$. This means that Mark will catch up to Andrew after 12.5 seconds.

  $$\begin{array}{rcl} 200 - 8x &=& 175 - 6x \\ +\,8x & & +\,8x \\ \hline 200 &=& 175 + 2x \\ -\,175 & & -\,175 \\ \hline \frac{25}{2} &=& \frac{2x}{2} \\ 12.5 &=& x \end{array}$$

  Substitute the x-value into either equation for x; the first equation is used here. Calculate the value of the right side of the equation to get the y-value of the solution.

  $$y = 200 - 8(12.5)$$
  $$y = 200 - 100$$
  $$y = 100$$

  Determine that Mark will catch up to Andrew because the solution (12.5, 100) contains positive values. Mark will catch up to Andrew after 12.5 seconds when they are both 100 meters from the finish line.

Key Academic Terms:

linear equation, intersection, graph, variable, system of equations, substitution, equivalent, solution, algebraic, parallel lines, properties of equality, point of intersection, ordered pair

Additional Resources:

- Activity: Fixing the furnace
- Activity: Summer swimming
- Activity: The intersection of two lines
- Video: Mike and Padhu: bikes and trikes
- Video: Solving simultaneous linear equations
- Video: Thinkport | Solving linear equations

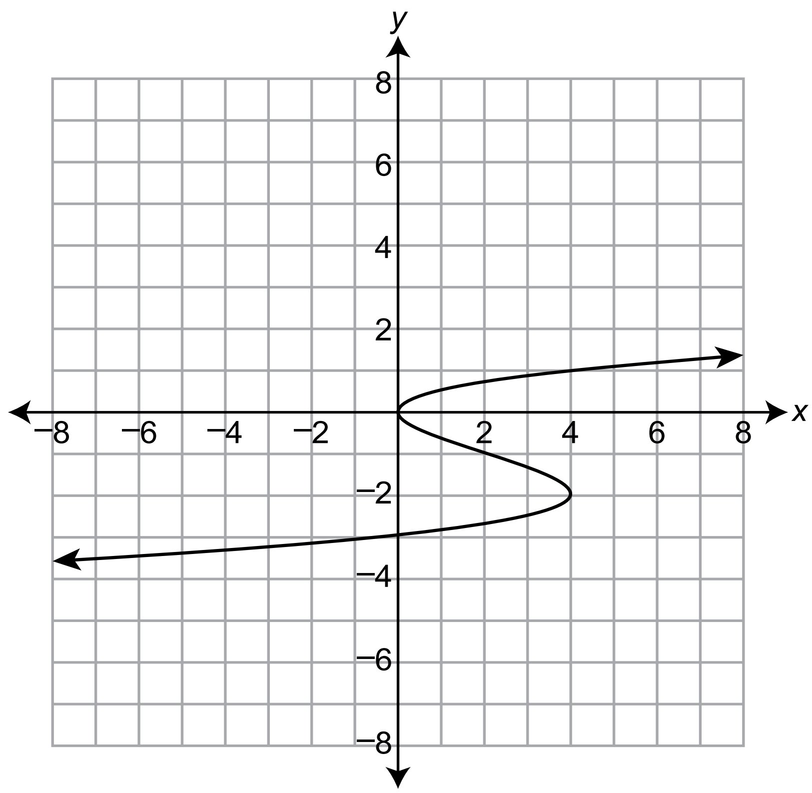
Algebra and Functions
Explain, evaluate, and compare functions.
13. Determine whether a relation is a function, defining a function as a rule that assigns to each input (independent value) exactly one output (dependent value), and given a graph, table, mapping, or set of ordered pairs.

Guiding Questions with Connections to Mathematical Practices:

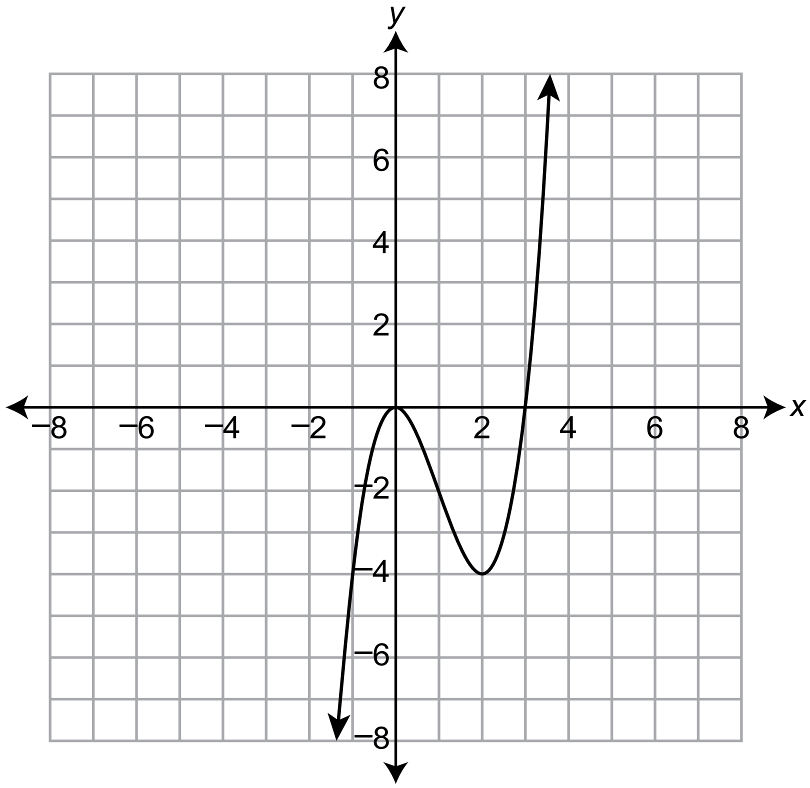
How can it be determined whether a table or a graph represents a function?

M.P.6. Attend to precision. Observe the table or graph carefully and note whether each input corresponds to only one output. For example, a table might have each input listed only once, or if it is listed more than once, the same output corresponds to it each time. Additionally, explain that because a vertical line on a graph corresponds to a single input value, if any vertical line intersects a graphed relation more than once, the relation is not a function.

- Ask students to examine relations given graphically, determine whether the relations represent functions, and give specific counterexamples for relations that are not functions. For example, given a graph, represent input values using vertical lines. For each input value, determine how many points have that input value by determining the number of times the corresponding vertical line intersects the relation. If the number of intersection points for any vertical line is greater than one, the relation is not a function. The following graph is not a function because some input values, like $x = 2$, have more than one output value.



The following relation is a function because every input value has at most one output value.



- Ask students to examine relations given in tables and determine if the relations represent functions. Students should be able to explain that the relation is a function if each input value has one and only one output value and give specific input values that do not fit these criteria for relations that are not functions. For example, the table shown represents a function because every input value has one and only one output value.

x	*y*
−1	3
0	2
1	3
2	6

The following table does not represent a function because the input value −1 is paired with more than one output value.

x	y
−1	3
0	4
−1	5
−2	6

Key Academic Terms:

input, output, function, graph, table, pattern, rule, proportional relationship, ordered pair, domain, range

Additional Resources:

- Activity: Functions
- Lesson: What is a function?
- Tutorial: Domain and range of functions
- Activity: The customers
- Activity: Introducing functions

Algebra and Functions
Explain, evaluate, and compare functions.
14. Evaluate functions defined by a rule or an equation, given values for the independent variable.

Guiding Questions with Connections to Mathematical Practices:

What does it mean to evaluate a function?

M.P.6. Attend to precision. Know that a function is evaluated when a particular value is substituted for a variable within an expression. For example, the function $y = 2x + 1$ can be evaluated when x equals 5 by substituting the number 5 in for the variable x. In this case, the resulting expression is $y = 2(5) + 1$, which gives $y = 11$. Additionally, know that a function can be evaluated for many different values for the variable(s).

- Provide students with a function. Ask them to evaluate it for several different values. For example, ask students to evaluate the function $y = -\frac{1}{3}x + 3$ when x equals −6, −3, and 3. When the function is evaluated for $x = -6$, the result is $y = 5$. When the function is evaluated for $x = -3$, the result is $y = 4$. When the function is evaluated for $x = 3$, the result is $y = 2$.
- Provide students with several different functions. Ask them to determine which one will produce the greatest or least result when evaluated for a particular value of x. For example, provide students with the following functions. Ask them to determine which will have the greatest result and the least result when evaluated for $x = -4$.
    - $y = \frac{1}{4}x + 4$
    - $y = -\frac{1}{4}x + 4$
    - $y = \frac{1}{4}x - 4$
    - $y = -\frac{1}{4}x - 4$

  In this case, the equation $y = -\frac{1}{4}x + 4$ will produce the greatest result, 5, when evaluated for $x = -4$. The equation $y = \frac{1}{4}x - 4$ will produce the least result, −5, when evaluated for $x = -4$.

What is an independent variable and how is it distinguished from a dependent variable?

M.P.6. Attend to precision. Know that the independent variable of a function is associated with the input values, while the dependent variable is associated with the output values. For example, in the function $y = 2.5x - 3$, the independent variable is x and the dependent variable is y. Additionally, know that an input-output table can be created to show particular values for the independent variable and the corresponding values for the dependent variable.

- Provide students with several different functions that are expressed as equations. Ask them to identify the independent and dependent variables in each equation. For example, ask students to identify the independent and dependent variables in the following functions:
  - $y = \frac{1}{10}x + \frac{1}{100}$
  - $p = \frac{1}{8}q$
  - $n = 5 - 5.5m$
  - $3.25k + 1 = j$

  Students should be able to identify the independent variables as x, q, m, and k and the dependent variables as y, p, n, and j.

- Provide students with a function. Ask them to create a corresponding input-output table for at least four different input values. For example, provide students with the function $m = -3p + 2$. Ask them to create an input-output table with the input values −2, −1, 1, and 2. Students should be able to create an input-output table similar to the one shown.

p	$-3p + 2$	m
−2	$-3(-2) + 2$	8
−1	$-3(-1) + 2$	5
1	$-3(1) + 2$	−1
2	$-3(2) + 2$	−4

  Since values for p were used to generate the values of m, p is the independent variable and m is the dependent variable.

Key Academic Terms:

input, output, function, graph, table, pattern, rule, proportional relationship, ordered pairs, independent variable, dependent variable, non-proportional, domain, range

Additional Resources:

- Activity: Functions
- Lesson: What is a function?
- Tutorial: Domain and range of functions
- Activity: The customers
- Activity: Introducing functions

Algebra and Functions
Explain, evaluate, and compare functions.
15. Compare properties of functions represented algebraically, graphically, numerically in tables, or by verbal descriptions. a. Distinguish between linear and non-linear functions.

Guiding Questions with Connections to Mathematical Practices:

How can properties of functions with different representations be compared?

M.P.1. Make sense of problems and persevere in solving them. Determine and explain correspondences between equations, verbal descriptions, tables, and graphs. For example, the initial value of a function, the output when the input is zero, can be found in any of the forms and is helpful when comparing functions. Additionally, compare and contrast attributes of functions of the same function family given in different representations.

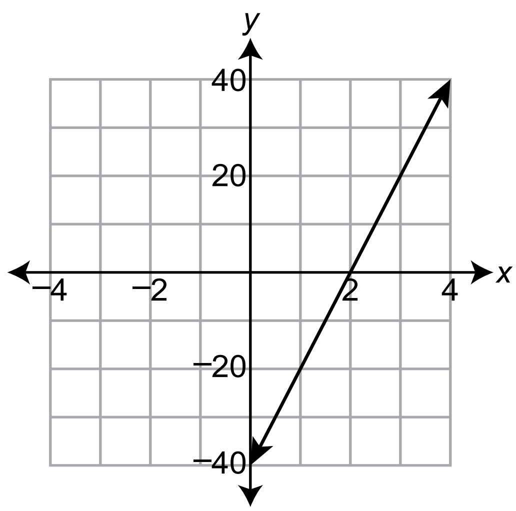
- Ask students to determine the distance between the y-intercepts of two functions given in different representations. For example, give students the equation $6x - 4y = 24$ and the following table that represents a different linear function.

x	y
−1	−14
1	−10
3	−6
5	−2

Determine that the y-intercept of the equation is (0, −6) by substituting a 0 in for x in the equation, meaning that when x is 0, the value of y is −6. Determine that the y-intercept of the function in the table is (0, −12) because the rate of change from the table is 2, so the output when the input is zero is −12. The distance between these y-intercepts is 6 units.

- Ask students to determine which linear function has the greatest slope when given several functions represented in different forms. For example, give students the equation $^{-}60x + 2y = 12$, a verbal description such as "Miles has \$45 and saves an additional \$50 every 2 weeks," the following table, and the graph below the table.

x	y
−2	−4
0	26
2	56
4	86



Determine the function with the greatest slope by comparing the rates of change between the four functions given in different forms. The equation has a slope of 30, the verbal description has a rate of change of \$25 per week, the table has a rate of change of 15, and the graph has a slope of 20, so the equation has the greatest slope when graphed.

How can it be determined whether a function is linear or not linear?

M.P.6. Attend to precision. Determine that nonlinear functions have a rate of change that varies between different pairs of points, which means their graphs are not lines. For example, the table of a function has the points (0, 0), (1, 1), (2, 4), and (3, 9). The rate of change between (0, 0) and (1, 1) is 1. The rate of change between (2, 4) and (3, 9) is 5. The rate of change is not the same between the pairs of points, so the function is nonlinear. Additionally, determine that the equations of linear functions follow certain patterns, while equations that do not represent linear functions do not follow the same patterns. Further, determine that functions that form straight lines on a graph are linear functions.

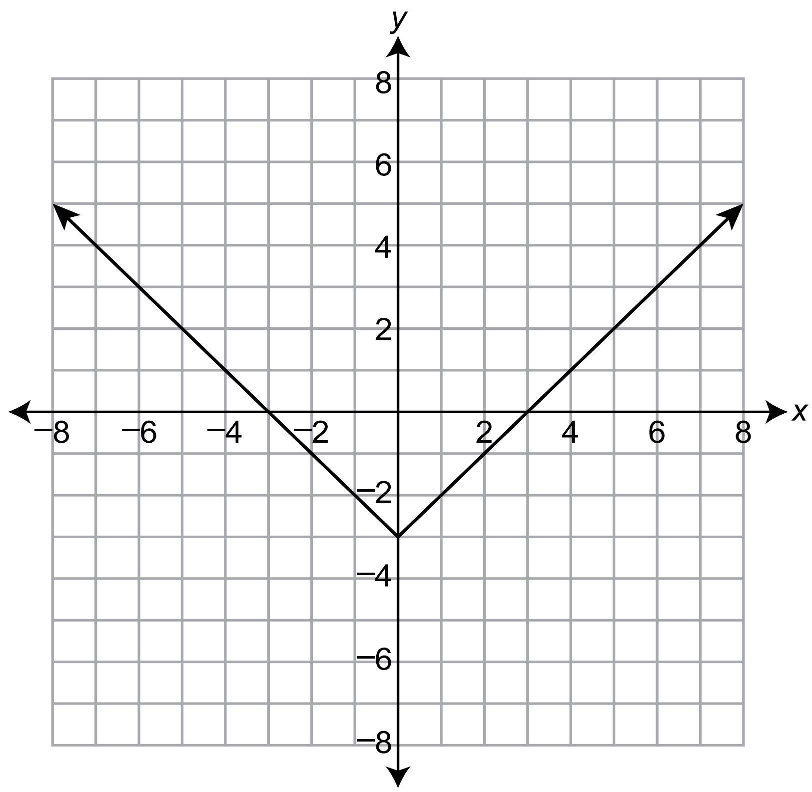
- Ask students to determine whether a function is linear or not linear when the function is given as an equation. For example, given the equation $2x + 3y = 6$, explain that the equation represents a linear function because there are no variables being multiplied together, division by variables, exponents other than 1 on variables, or operations on variables like absolute values or radicals. The equation $y = 4x^3 - 6$ is not linear because x is raised to an exponent other than 1.

- Ask students to determine whether a function is linear or not linear when the function is given as a set of points or a table. Then, ask students to find the rate of change between pairs of points and explain that because the rates of change are not the same, the function is not linear. For example, a table representing a function is given.

x	y
−4	8
−2	2
0	0
2	2

The rate of change between (−4, 8) and (−2, 2) is −3, the rate of change between (−2, 2) and (0, 0) is −1, and the rate of change between (0, 0) and (2, 2) is 1. The rates of change are not the same, so the function is not linear.

- Ask students to determine whether the graph of a relation represents a linear function. For example, given the graph, determine that while the graph represents a function because each input has only one output, it is not a straight line and therefore not a linear function.



Key Academic Terms:

input, output, function, graph, table, rate of change, initial value, verbal description, slope, *y*-intercept, nonlinear, side length, linear function, nonlinear function, constant rate of change, linear, domain, range

Additional Resources:

- Activity: Battery charging
- Video: Representing a direct variation algebraically and graphically
- Lesson: Turtle & Snail part I: An introduction to “Rule of Five”
- Lesson: Compare slopes of functions I
- Lesson: Compare linear relationships I
- Lesson: Linear vs quadratic
- Article: Teaching linear versus nonlinear functions through discovery
- Lessons: Walk the line: a module on linear functions
- Article: 10 activities to practice linear functions like a boss

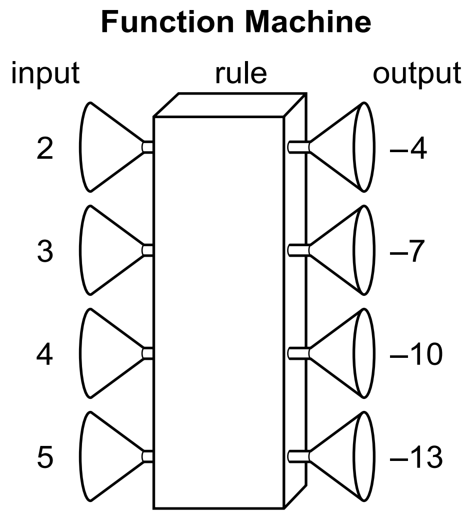
Algebra and Functions
Use functions to model relationships between quantities.
16. Construct a function to model a linear relationship between two variables. a. Interpret the rate of change (slope) and initial value of the linear function from a description of a relationship or from two points in a table or graph.

Guiding Questions with Connections to Mathematical Practices:

How do patterns, rules, and proportional relationships connect to functions?

M.P.2. Reason abstractly and quantitatively. Observe functions in terms of a pattern, rule, or proportional relationship to make sense of how the variables relate to each other. For example, skip-counting by 15 is a pattern that can be represented by $y = 15x$, where x is the number of each term in the pattern, y is the value of that term, and the rule is to skip-count by 15. Additionally, determine the function rule for simple functions given a set of inputs and outputs for a given function. Further, given real-world context, determine a function rule that can be used to represent the context.

- Ask students to write the function rule for a given table of input and output values. For example, in the graphic shown, the "Function Machine" shows that each y-value is 2 more than $^{-}3$ times the corresponding x-value.



This pattern can also be represented in a table of input and output values.

x	y
2	$^{-}4$
3	$^{-}7$
4	$^{-}10$
5	$^{-}13$

From the table, develop the rule $y = {}^{-}3x + 2$. This rule describes a function because for any input value, following the operations of multiplying by $^{-}3$ and adding 2 always yields the same output value. Two people correctly applying the rule will always come up with the same result.

- Ask students to write the function rule for a given real-world context. For example, given that a restaurant charges \$3.50 per hamburger, determine that the equation $y = 3.5x$ can be used to represent the cost, y, for x number of hamburgers. Further, any proportional relationship is always a function.

How can the rate of change and initial value be determined from a description of a relationship, two (x, y) values, a graph, or a table?

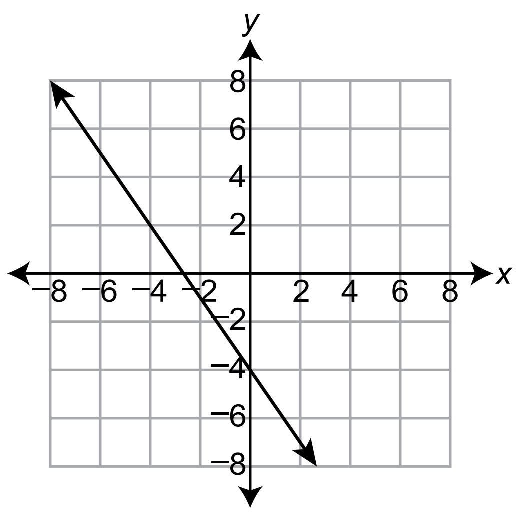
M.P.4. Model with mathematics. Make use of what is known about the relationship and the meanings of x and y in a function. For example, a line that passes through the points (3, −4) and (5, 2) changes by 6 in the y-value and by 2 in the x-value, making the ratio of the y-value to the x-value $\frac{6}{2} = \frac{3}{1}$. Then, use one of the given points and the rate of change, $m = 3$, to solve for the initial value, b, in $y = mx + b$. Additionally, given a verbal description containing two values of the independent variable and the corresponding values of the dependent variable, determine the rate of change by finding the ratio of the change in the dependent variable to the change in the independent variable. Then, use one of the values of the independent variable and the corresponding value of the dependent variable to determine the initial value using $y = mx + b$ where (x, y) is the value of the independent variable and the corresponding value of the dependent variable and m is the rate of change. Further, the rate of change can be determined on the graph of a line by finding the ratio of vertical change to horizontal change for any two points on the graph, and the initial value can be determined by determining the point at which the graph intersects the y-axis.

- Ask students to determine the rate of change and initial value given a linear relationship in a table. For example, the following table is given.

x	y
−3	9
−1	5
1	1
3	−3

Determine that the rate of change in the table is −2 by using any two points in the table and finding the change in the values of the dependent variable (y) divided by the change in the values of the independent variable (x) for the two points. Then, determine that the initial value or y-intercept of the line is (0, 3) by using the rate of change as m and one of the points from the table as (x, y) in $y = mx + b$ and solving for b, which represents the initial value or y-intercept.

- Ask students to determine the rate of change and initial value given a verbal description of a linear relationship that gives two values of the independent variable and the corresponding values of the dependent variable. For example, the following situation is given: "At a bowling alley, the cost to rent a lane is a linear relationship. It costs \$17.25 to rent a lane at the bowling alley for $1\frac{1}{2}$ hours and it costs \$21 to rent a lane for 2 hours." Determine that the hourly cost to rent a lane is \$7.50 because the difference in the given costs is \$3.75 and the difference in time is $\frac{1}{2}$ hour, so the unit rate is $\frac{3.75}{\frac{1}{2}} = \frac{7.5}{1}$ dollars per hour. Determine the initial cost to rent a lane to be \$6, because for 2 hours the total cost is \$21, and the unit rate times 2 hours is \$15, leaving a difference of \$6.
- Ask students to determine the rate of change and initial value given the graph of a linear relationship. For example, the graph shown is given.

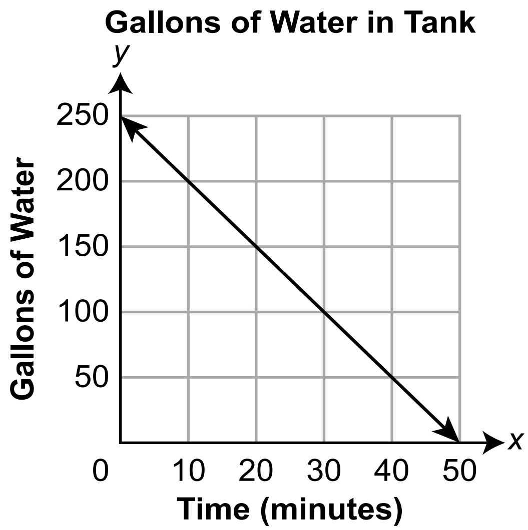


Determine that the rate of change is $-\frac{3}{2}$ by identifying two points on the graph such as (−2, −1) and (0, −4). Determine that the initial value of the function is −4 because the graph contains the point (0, −4).

What do the rate of change and initial value represent in a given context?

M.P.2. Reason abstractly and quantitatively. Interpret the meaning of the rate of change and initial value for a context and the table or graph of that context. For example, a scenario where someone earns $15 per week in allowance and starts with $50 in savings would result in an initial value of $50 and a rate of change of $15 per week, so the graph would start at (0, 50) and increase by 15 for each weekly increment. Additionally, interpret the meaning of the rate of change and initial value for a context given an equation that represents the context. For example, given that the equation $y = -12x + 25$ represents the distance in miles, y, that Rowan is away from home after riding his bike for x hours, determine that Rowan started 25 miles from home and is riding at a rate of 12 miles per hour toward home.

- Ask students to construct a graph to represent a given context. For example, given that the amount of water in a rainwater tank starts at 250 gallons and the amount of water is decreasing at a rate of 5 gallons per minute, construct the graph by plotting the initial value on the y-axis and using the rate of change to plot another point such as (1, 245) or (2, 240).



- Ask students to construct a table to represent a given context. For example, given that someone who is selling boxes of cookies has already sold 12 boxes and will sell an additional 3 boxes per day, construct the table by adding the initial value to the rate of change times the input value.

Number of Days	Boxes of Cookies Sold
1	15
2	18
3	21
4	24

- Ask students to interpret the meaning of the rate of change and initial value for a context given an equation that represents the context. For example, given that the equation $y = 2.5x + 4$ represents the weight of a puppy in pounds, y, based on the puppy's age in months, x, determine that the puppy was 4 pounds when it was born because when x is 0, y is 4. Then, determine that the puppy is growing at a rate of 2.5 pounds per month because the puppy's age in months is being multiplied by 2.5, so when x increases by 1, y increases by 2.5.

How can a function be constructed to model a linear relationship between two quantities?

M.P.4. Model with mathematics. Look for the features of a function, such as rate of change (m) and the point $(0, b)$, also called "the y-intercept," within a relationship. For example, the input is always the independent variable and the output is always the dependent variable. The rate of change is the change in the dependent variable (y) divided by the change in the corresponding independent variable (x). The function is constructed by using the rate of change, initial value, and variables to show the relationship between the quantities. Additionally, when two points on the graph of a linear function are known, the function can be represented as an equation by finding the rate of change between the points and then writing the equation in the form $y - y_1 = m(x - x_1)$ where m is the rate of change and (x_1, y_1) is one of the given points. Further, know that the equation can also be written in the form $y = mx + b$.

- Ask students to write an equation for a linear relationship given any two points on the graph of the function by finding the rate of change between the points and then writing the equation in the form $y - y_1 = m(x - x_1)$ where m is the rate of change and (x_1, y_1) is one of the given points. For example, given that the graph of a linear function contains the points $(-2, 5)$ and $(2, 11)$, determine that the rate of change between the points is $\frac{6}{4} = \frac{3}{2}$ by finding the change in the dependent variable (y) divided by the change in the independent variable (x). The equation can be written as either $y - 5 = \frac{3}{2}(x + 2)$ or $y - 11 = \frac{3}{2}(x - 2)$, and either equation is equivalent to $y = \frac{3}{2}x + 8$.

- Ask students to write an equation for a linear relationship given the y-intercept and another point on the graph of the function. For example, given that the y-intercept of a linear function is $(0, -3)$ and the point $(-4, -1)$ is also on the graph of the function, determine that the rate of change between the points is $-\frac{2}{4} = -\frac{1}{2}$ by finding the change in the dependent variable (y) divided by the change in the independent variable (x). As a result, determine that the equation for the function is given by $y = -\frac{1}{2}x - 3$.

Key Academic Terms:

input, output, function, linear, initial value, rate of change, independent variable, dependent variable, *y*-intercept

Additional Resources:

- Lesson: Slope & rate of change
- Lesson: Compare slopes of functions I
- Activity: Downhill
- Lesson: Using linear functions for modeling
- Lessons: Walk the line: a module on linear functions
- Article: 10 activities to practice linear functions like a boss

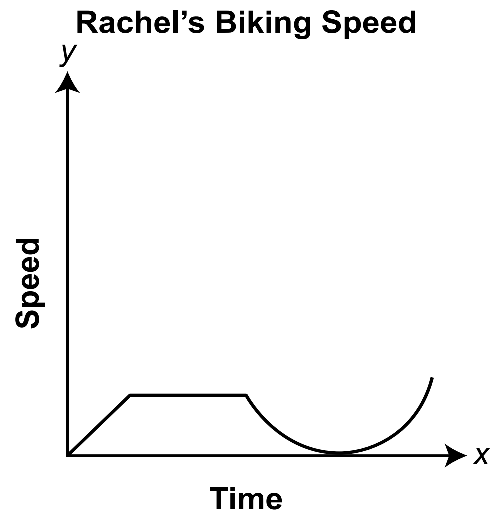
Algebra and Functions
Use functions to model relationships between quantities.
17. Analyze the relationship (increasing or decreasing, linear or non-linear) between two quantities represented in a graph.

Guiding Questions with Connections to Mathematical Practices:

How can analyzing a graph qualitatively help determine the functional relationship between two quantities?

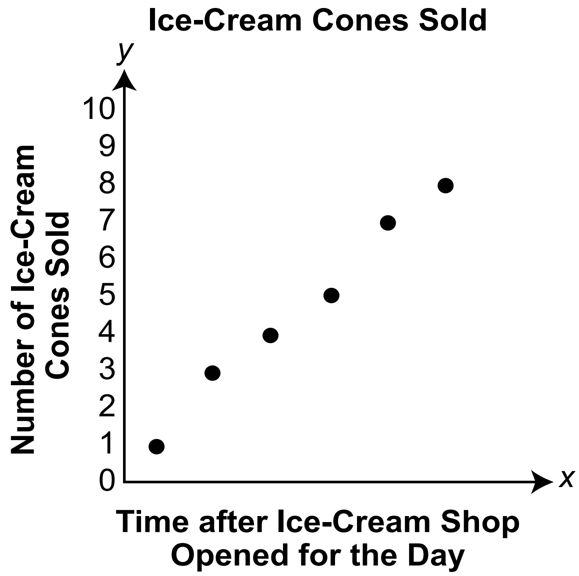
M.P.1. Make sense of problems and persevere in solving them. Analyze a graph using its features and general shape, reading from left to right to describe what happens to the output as the input increases. For example, an increasing linear graph comparing time to distance shows that the rate of change is the same for every interval, so the speed is constant. Additionally, know that the horizontal axis is used to represent the independent variable, and as the independent variable increases from left to right, the dependent variable can increase, decrease, or remain constant. Further, observe that the context of the independent and dependent variables of a function determines whether the graph of the function is discrete or continuous. For example, a graph of distance versus time would be continuous, but a graph of visitors to a park during different days is discrete because the number of people can only be represented using whole numbers.

- Ask students to describe where a graph shows a functional relationship that is increasing, decreasing, or constant. On intervals that the graph shows an increase or decrease, determine whether that increase or decrease is linear or nonlinear. For example, give students the graph shown.



Rachel increased her speed on her bike at a linear rate, then she rode at a constant speed for a while before slowing down to a stop at a nonlinear rate. Rachel then continued to ride her bike by increasing her speed at a nonlinear rate.

- Ask students to explain why a certain relationship is represented using a continuous or discrete graph by using the independent and dependent variables given on the graph and the appropriate set of numbers that best represents each variable given. For example, give students the graph shown.

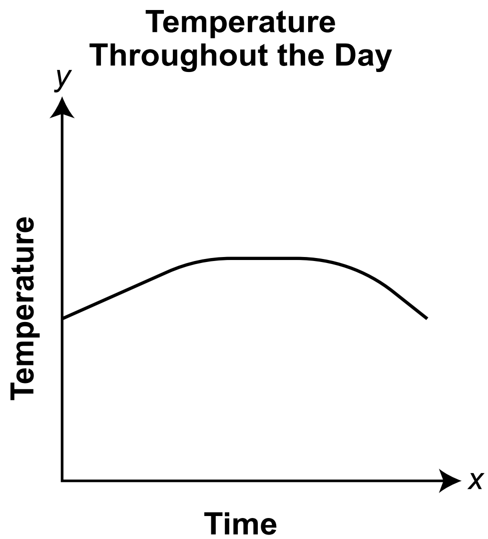


Explain that the graph is discrete because the number of ice-cream cones sold can only be represented using whole numbers, as shown, while time can be represented using all nonnegative real numbers.

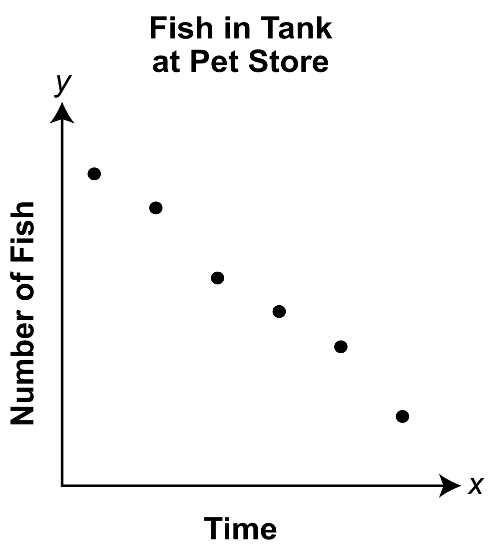
How can the qualitative features of the verbal descriptions of a function be represented in a graph?

M.P.4. Model with mathematics. Represent a function with a sketched graph to show the relationship between the two quantities. For example, to show a bank account that starts with a large amount of money and decreases by varying small amounts of money each month for fees and withdrawals, use a decreasing nonlinear graph. Additionally, determine the independent and dependent variables in a given verbal description, assigning the independent variable to the horizontal axis on the graph and the dependent variable to the vertical axis on the graph. For example, given that a car slows down before driving at a steady speed, determine that the horizontal axis represents time and the vertical axis represents speed when sketching the graph of the function. Further, determine when a function given as a verbal description is best represented using a continuous or a discrete graph.

- Ask students to sketch the graph of a function given a verbal description of the function. For example, given that the temperature rises slowly in the morning, becomes constant through the day, and then decreases rapidly after the sun sets, sketch a graph. One possible graph is shown.



- Ask students to reason whether a given verbal description is best modeled on a graph by a continuous or discrete graph before graphing the function. For example, given that the number of fish in a tank at a pet store decreased throughout the day as customers purchased them, explain that a discrete graph would best model the situation because the number of fish should not be modeled using all real numbers, but only the whole numbers. Sketch a graph similar to the one given.



Key Academic Terms:

input, output, function, linear, nonlinear, interval, rate of change, constant speed, qualitative, sketch, continuous, discrete, y-intercept, x-intercept

Additional Resources:

- Activity: The Lowdown | Functional relationships between quantities: calculating fuel consumption
- Activity: Tides
- Activity: Bike race
- Activity: Riding by the library
- Lesson: Using linear functions for modeling
- Lessons: Walk the line: a module on linear functions
- Article: 10 activities to practice linear functions like a boss

Data Analysis, Statistics, and Probability
Investigate patterns of association in bivariate data.
18. Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities, describing patterns in terms of positive, negative, or no association, linear and non-linear association, clustering, and outliers.

Guiding Questions with Connections to Mathematical Practices:

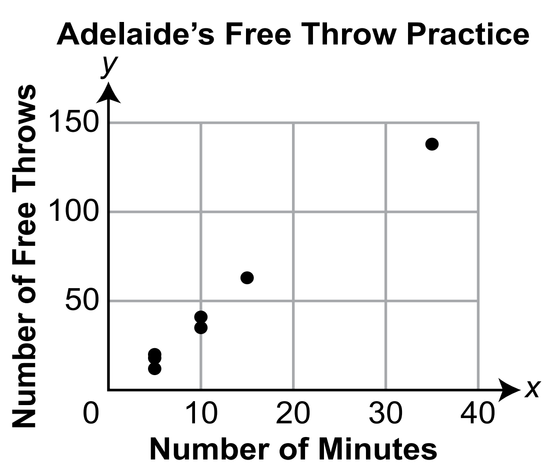
How are graphs for bivariate data constructed?

M.P.4. Model with mathematics. Interpret the two quantities as the x- and y-variables on a coordinate grid to construct a scatter plot. For example, the daily high temperatures during the month of August can be represented with x-values while the number of frozen desserts sold at a concession stand each day in August can be represented with y-values. Additionally, note that either quantity can be used as the x-variable, though often one of the variables makes more sense, depending on the context of the data.

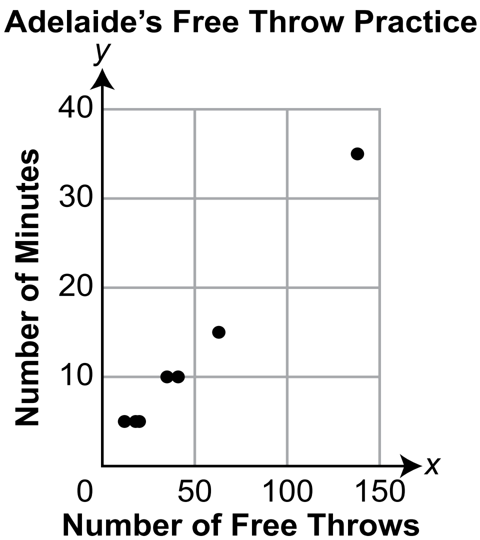
- Ask students to create a scatter plot from a given set of bivariate data. For example, the number of minutes that Adelaide spent practicing basketball free throws and the number of free throws she shot each day last week is shown in the table.

Number of Minutes	10	5	35	5	15	10	5
Number of Free Throws	35	18	138	12	63	41	20

Interpreting the number of minutes spent practicing as the x-variable gives the following scatter plot.



Alternatively, students may interpret the number of free throws as the *x*-variable. This interpretation gives the following scatter plot.



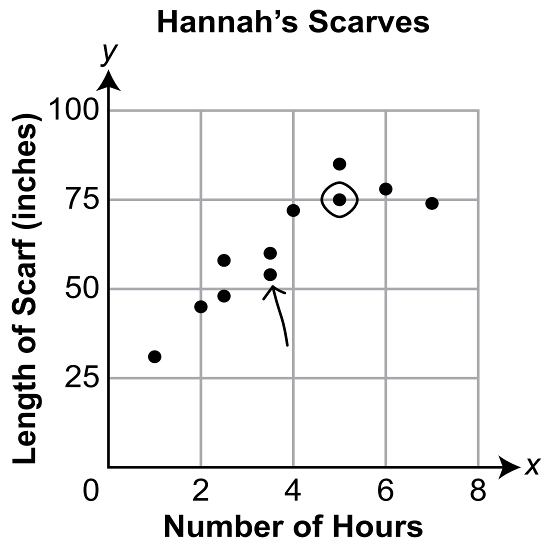
- When provided with two related variables, ask students to determine which is the dependent (*y*) and which is the independent (*x*) variable. For example, give students the following pairs of variables:
  - the average time visitors spend on a website and the size of the font used on the site

    The owners of a website would be interested in knowing if the size of the font affects how long visitors spend on the website. However, visitors spending more time on the website is not going to affect the font size. It makes more sense to use the font size as the independent x-variable.

  - the amount of horsepower a car's engine has and the top speed of the car

    Depending on the context, either variable could make sense as the independent variable. A person considering buying a car might use the advertised horsepower to predict the top speed of the car. In that case, that person would use the horsepower as the independent variable. However, a company that designs race cars could have a target speed in mind. They might want to know how much horsepower a car would need in order to reach that speed. In that case, the company would use the top speed as the independent variable.

- Ask students to interpret data that is presented in a scatter plot. For example, show students the following scatter plot about scarves that Hannah knits.

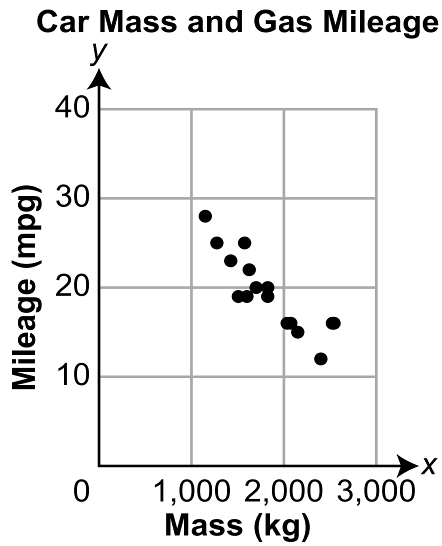


Help students to identify that the circled point indicates that a 75-inch scarf took Hannah about 5 hours to knit and that the two points indicated above the arrow represent scarves that each took Hannah about 3.5 hours to knit, but the scarves were different lengths.

How do scatter plots show the relationship between two quantities?

M.P.2. Reason abstractly and quantitatively. Observe a scatter plot to determine any patterns that occur and interpret the meaning of those patterns within the given context. For example, a scatter plot that shows Julia's age in years on the x-axis and Julia's height in inches on the y-axis will reveal the pattern that as Julia gets older, she generally grows taller, up to a certain age, and then her height remains constant. Additionally, observe that a scatter plot can also show a lack of relationship between two quantities.

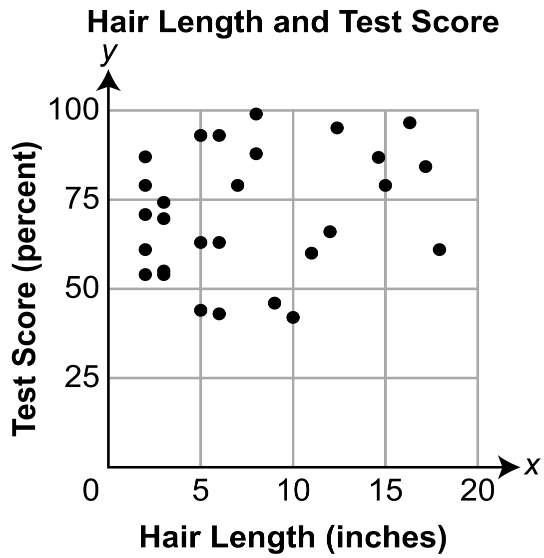
- Ask students to describe how changes in the x-variable of a given scatter plot affect the y-variable of that plot. For example, the mass and gas mileage of various cars are shown in the following scatter plot.



  Help students to observe that in general, as the mass of a car increases, the gas mileage decreases. Further, the gas mileage decreases at a fairly constant rate. A 100-kilogram increase from 1,500 kilograms to 1,600 kilograms decreases the gas mileage by about the same amount as an increase from 2,200 kilograms to 2,300 kilograms.

- Ask students to predict and describe what a scatter plot of two given variables would look like. For example, ask students to describe a scatter plot of people's scores in a game of bowling and the number of times people have bowled previously. In general, as people bowl more games, their scores will increase. However, this increase will be capped at the maximum score of 300 points, which means it will level off and will not change at a constant rate.

- Ask students to explain how to recognize when a scatter plot shows no relationship between two variables. For example, a scatter plot of students' hair lengths and the scores they received on a math test is shown.

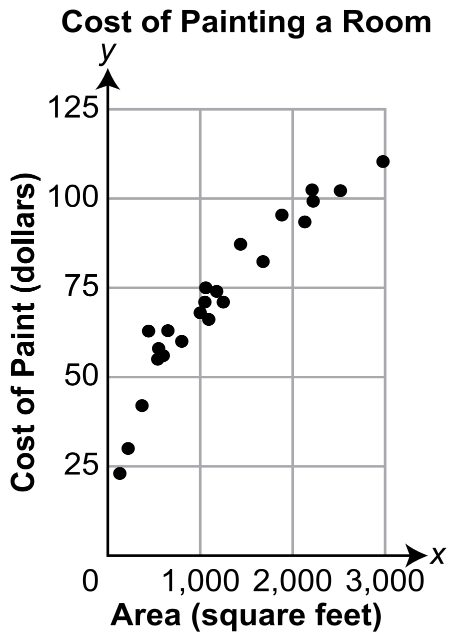


This scatter plot shows no relationship because students with different hair lengths all have approximately the same spread of test scores. Likewise, students with different test scores all have approximately the same spread of hair lengths.

How can scatter plots of bivariate data be described?

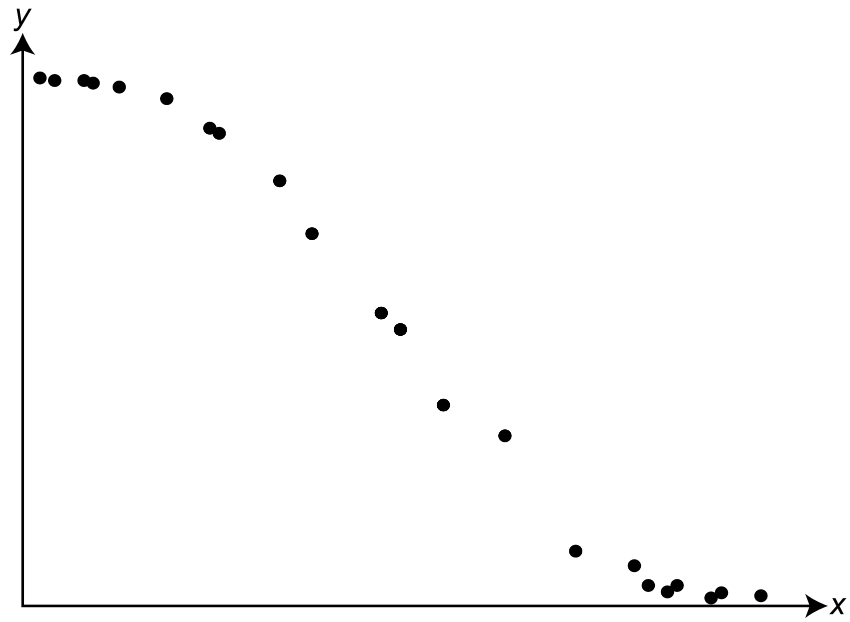
M.P.6. Attend to precision. Represent patterns in a scatter plot with terms such as clustering, outliers, positive or negative association, linear association, and nonlinear association. For example, a scatter plot can be described as having a positive linear association or a negative linear association, and it may also have some clustering and outliers, but the scatter plot cannot have both a positive association and a negative association. Additionally, predict whether a given context will have a positive or negative association.

- Ask students to determine whether a given scatter plot shows a positive or negative association, linear or nonlinear association, and outliers. For example, a painting company creates a scatter plot of the area of a room versus the cost of the paint needed. The scatter plot is shown.



The plot shows a positive association between the variables because as the area of a room increases, the cost of paint for the room also increases. The association is nonlinear because it shows a clear curve. There are no clear outliers in this data set.

- Ask students to sketch a scatter plot that fits a given set of criteria. For example, ask students to sketch a scatter plot that shows a negative nonlinear association between its variables.



The plot shows a negative association between the variables because as the *x*-value increases, the *y*-value decreases. The association is nonlinear because it shows a clear curve and is not a straight line.

- Give students a description of how two abstract variables are related and ask students to find examples of contexts that fit the given description. For example, ask students to find examples of two variables that have a positive and linear association. Some possible student responses include:
  - arm span versus height of humans
  - total gallons of gas sold in a day versus money earned from gas sales that day at a gas station

Key Academic Terms:

scatter plot, bivariate, data, clustering, positive association, negative association, outlier, linear association, nonlinear association, univariate, no association, ordered pair, independent variable, dependent variable, axis, pattern

Additional Resources:

- Lesson: Shake it up with scatterplots
- Article: 11 activities that make practicing scatter plot graphs rock
- Activity: Patterns of association—quality of English spoken by people who speak Spanish in their homes
- Video: Bungee jump
- Activity: Scatter plot
- Activity: Animal brains

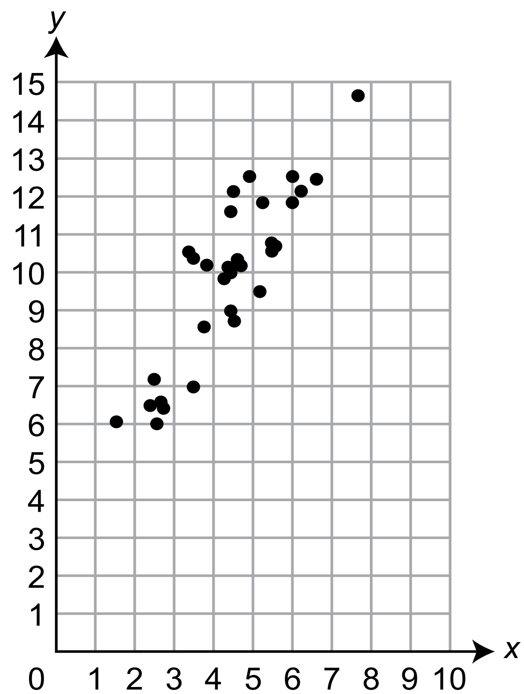
Data Analysis, Statistics, and Probability
Investigate patterns of association in bivariate data.
19. Given a scatter plot that suggests a linear association, informally draw a line to fit the data, and assess the model fit by judging the closeness of the data points to the line.

Guiding Questions with Connections to Mathematical Practices:

When can a relationship between two quantitative variables on a scatter plot be modeled by a straight line?

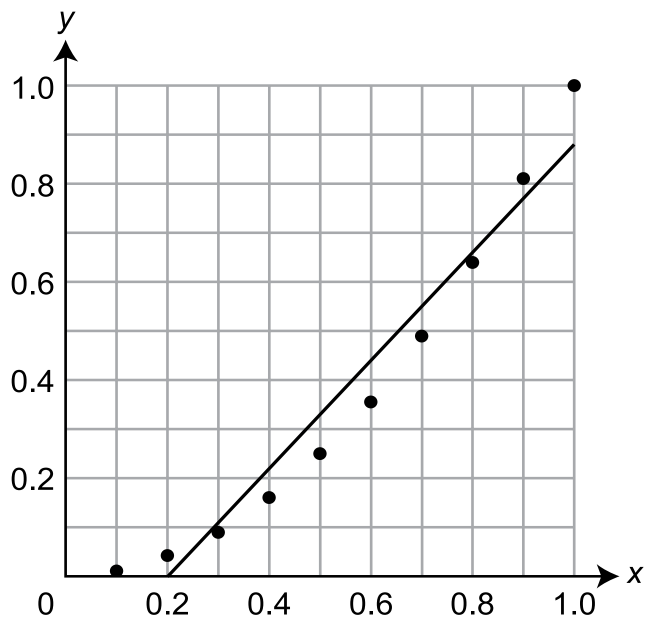
M.P.4. Model with mathematics. Represent a scatter plot with a line to fit the data when there is a clear linear association between the two variables. For example, a scatter plot with a negative linear association would have a line with a negative slope that comes close to most of the data points on the graph. Additionally, know that even if a straight line fits a set of data well, a curved line might sometimes be more appropriate.

- Ask students to determine from a given scatter plot whether there seems to be a linear association between the two variables and to justify their decision by explaining that a straight line can be drawn that comes close to most of the data points. For example, give students the scatter plot shown.



The data seem to show a linear association because there is a clear increasing trend of an approximately constant rate and a straight line can be drawn that comes close to most of the data points.

- Ask students to determine whether a curved graph might be more appropriate than a straight line even in situations when a straight line is reasonable. For example, give students the scatter plot shown.

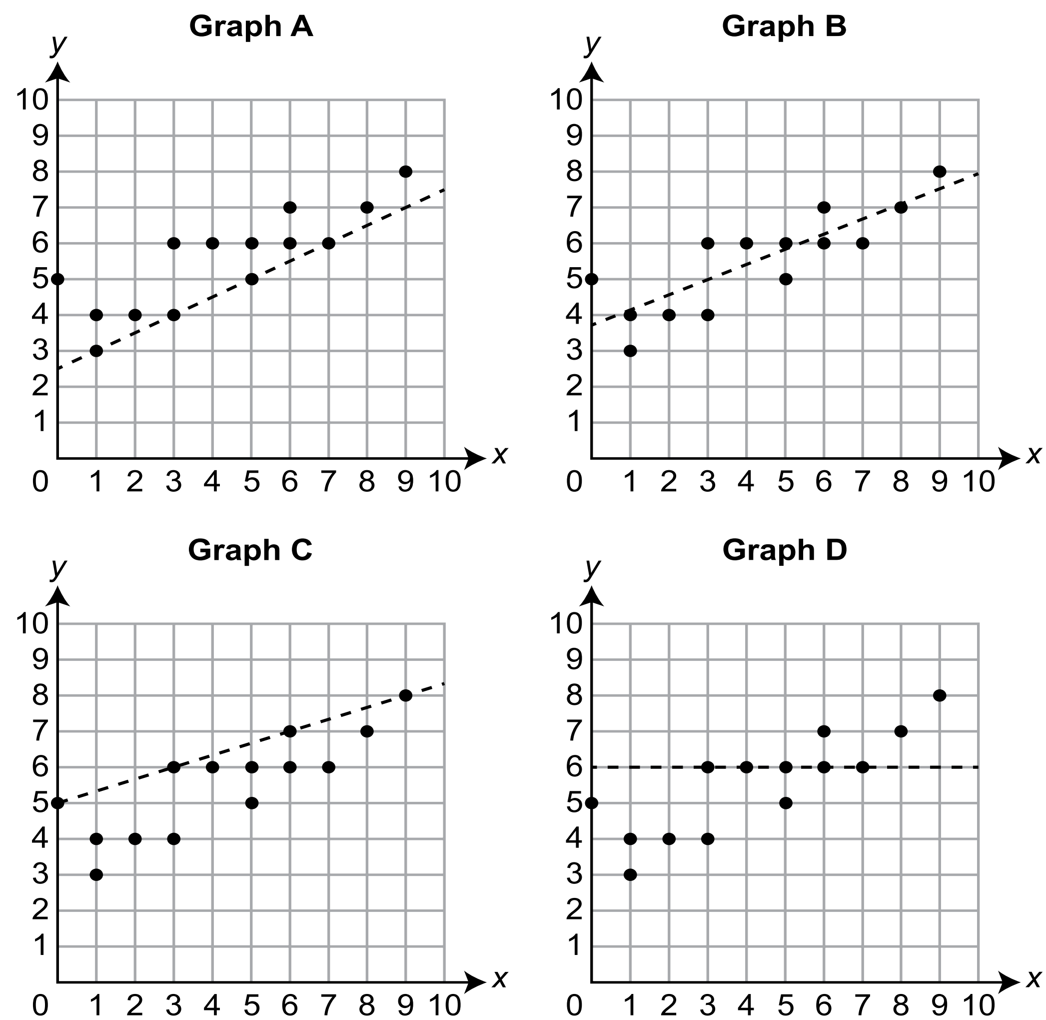


The straight line follows the upward trend of the points, and the line is close to all of the points. However, all the points below the line are in the middle, and all the points above the line are at the ends. This pattern suggests that the points follow a slight curve and that a curved graph might be more appropriate.

How can a line be graphed on a scatter plot to best show the relationship between two variables?

M.P.8. Look for and express regularity in repeated reasoning. Find a line with an approximate slope and y-intercept to fit the scatter plot data points. For example, lay a string on top of a scatter plot and keep the string as close to all the data points as possible to visualize a line that fits the data. Additionally, know that a line that is somewhat close to all the data points is considered to be better than a line that is very close to most of the data points but very far from a few data points.

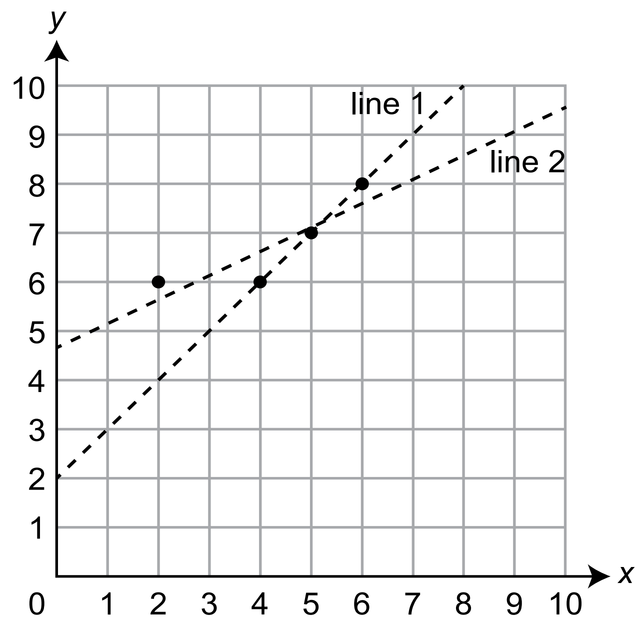
- Give students a scatter plot along with several lines and ask them to discuss the characteristics that make each line a better fit or a worse fit. For example, the same scatter plot is shown with four different lines.



  - ○ The line in graph A passes perfectly through four points and follows the general increasing trend of the points, but it is clearly below most of the points.
  - ○ The line in graph B lies in the middle of the cluster of points and follows the general increasing trend of the points; however, it does not pass perfectly through any of the points.
  - ○ The line in graph C passes perfectly through four points and follows the general increasing trend of the points, but it is clearly above most of the points.
  - ○ The line in graph D passes perfectly through five points and passes through the middle of the cluster of points, but it does not follow the general increasing trend of the points.

Ask students to discuss the merits of each line. Help them to know that in order to best show a relationship, a line should pass through the middle of the cluster and should follow the trend of the points, regardless of how many points it passes through. Conclude that graph B best models the relationship.

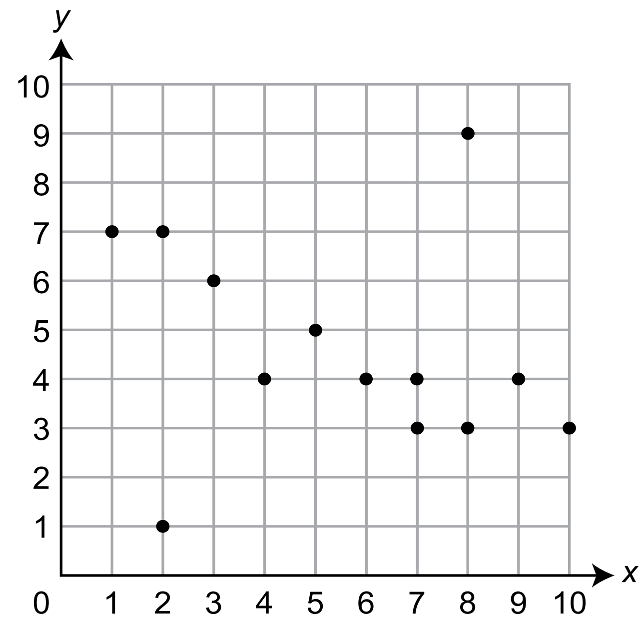
- Ask students to discuss an example with more than one possible line to fit the data. For example, show students the following scatter plot.



Students should conclude that line 2 is a better representation of the data. It is better to be close to all points but not exact (as with line 2) than to be exact on most points but far from one (as with line 1).

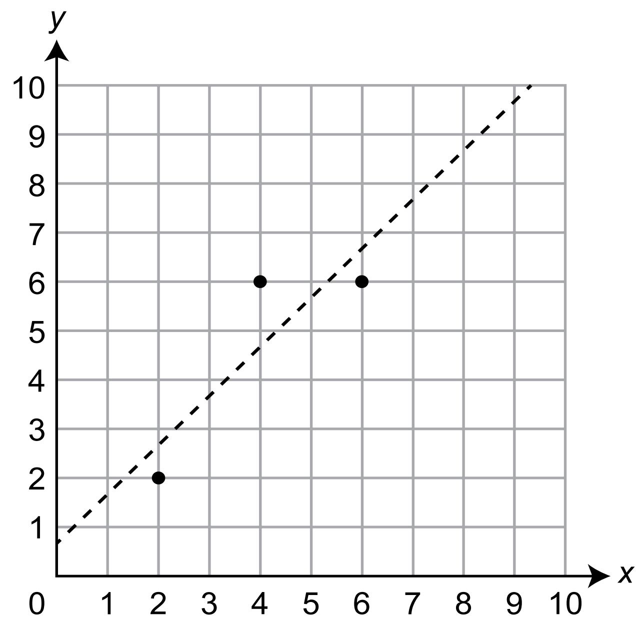
M.P.6. Attend to precision. Analyze the effect that outliers have when fitting a line to data and adjust the line according to the outlier. For example, notice that an extreme outlier of a data set will have a more significant effect on the fit of a line than an outlier that is less extreme. Additionally, demonstrate that a point that is far from the other points but still follows the same trend will not significantly change the fit of a line.

- Give students a scatter plot and ask them to identify points that are outliers. For example, show students the following scatter plot.

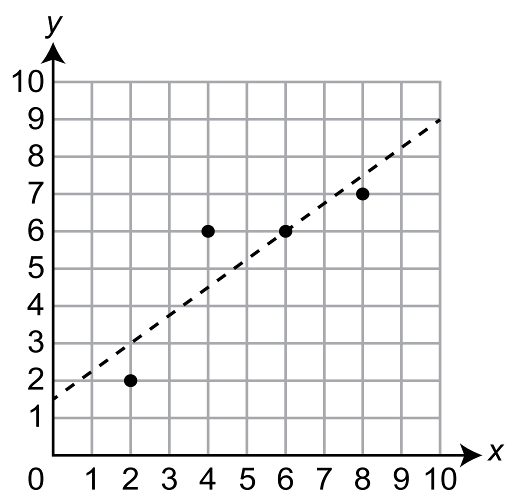


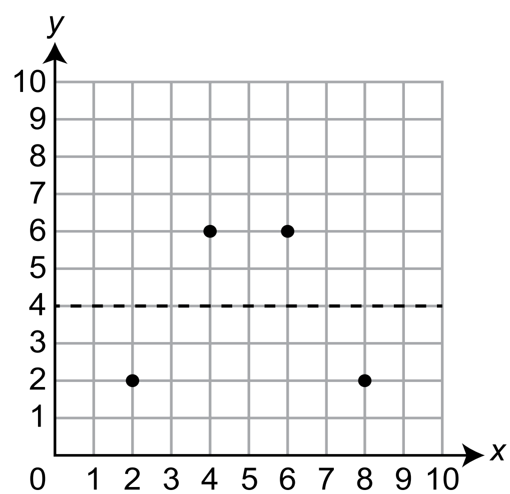
Students should identify the points at (2, 1) and (8, 9) as outliers.

- Give students a scatter plot with the line to fit the data shown. Then, give them the same scatter plot with one additional point plotted and ask them to estimate a new line to fit the data. Give some students a plot where the additional point is an outlier and give some students a plot where the additional point is not an outlier. Ask students to compare their graphs and discuss the results. For example, give students the following scatter plot.



Then, give students the same scatter plot but with a new point added. Two possible scatter plots are shown with possible student estimates of a line to fit the data.





The line for the first graph is much closer to the original line because the additional point is not an outlier and follows the trend of the original set of data. The line for the second graph is much more different from the original line because the additional point is an outlier compared to the original three points.

Key Academic Terms:

scatter plot, bivariate, data, clustering, association, outlier, line of best fit, quantitative, ordered pair

Additional Resources:

- Article: 11 activities that make practicing scatter plot graphs rock
- Lesson: Shake it up with scatterplots
- Activity: Patterns of association—quality of English spoken by people who speak Spanish in their homes
- Video: Bivariate data and analysis: anthropological studies
- Video: Bungee jump
- Activity: Scatter plot
- Activity: Animal brains

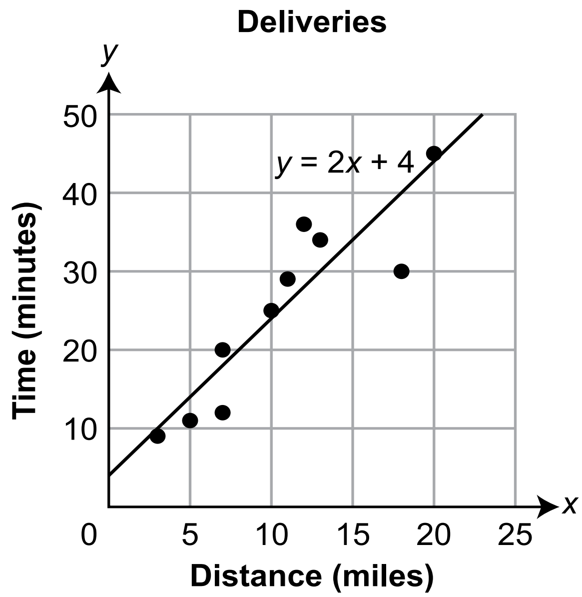
Data Analysis, Statistics, and Probability
Investigate patterns of association in bivariate data.
20. Use a linear model of a real-world situation to solve problems and make predictions. a. Describe the rate of change and y-intercept in the context of a problem using a linear model of a real-world situation.

Guiding Questions with Connections to Mathematical Practices:

How can the equation of a linear model help solve problems in the context of bivariate data?

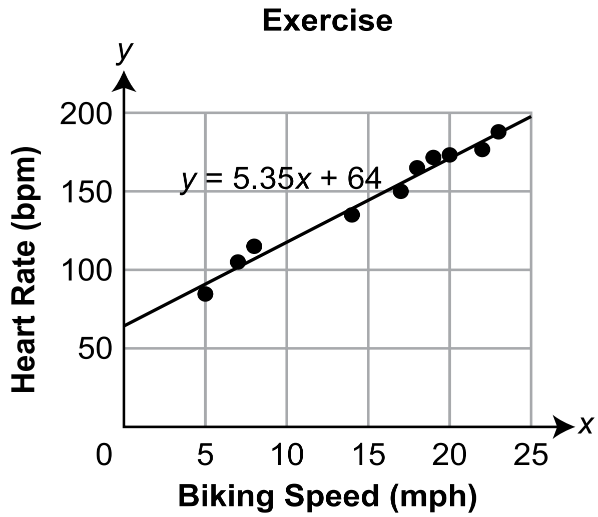
M.P.4. Model with mathematics. Interpret the rate of change and y-intercept of a fitted line in context to better interpret bivariate data and solve problems. For example, if a scatter plot with fitted line $y = -8x + 29$ models the amount of water, in milliliters, in a container after x days, then the data have a decreasing association because the rate of change, or slope, of the line is -8, which means that the amount of water is decreasing by approximately 8 milliliters per day. The container started with 29 milliliters of water, so the initial value is 29 because when $x = 0$, the y-intercept is 29. Additionally, know that the y-values given by a linear model are predicted values, not necessarily actual values.

- Ask students to interpret the rate of change and the y-intercept of a linear model of the data. For example, a delivery driver keeps track of his distance driven and the time needed for several deliveries, plots his data, and estimates a line to fit the data, as shown.



Ask students to explain the significance of the 2 and the 4 from the equation in context. Each additional mile increases the estimated delivery time by about 2 minutes, and the time before a delivery can start is 4 minutes.

- Ask students to use the equation of a linear model to make predictions based on data. For example, when Cory exercises, he tracks his biking speed in miles per hour and his heart rate in beats per minute. His data and a linear model are shown.



Use the equation of the model to predict Cory's heart rate if he bikes at a speed of 10 miles per hour. The speed of 10 miles per hour can be substituted into the equation to obtain a prediction of $y = 5.35(10) + 64 = 117.5$ beats per minute. Be sure to emphasize that this is a prediction rather than a precise value. In reality, if Cory bikes at a speed of 10 miles per hour, his heart rate can be expected to be close to 117.5 bpm, but probably not exactly 117.5 bpm. This is because the line merely models the trend of the data and not each exact data point.

Key Academic Terms:

scatter plot, bivariate, data, clustering, increasing association, decreasing association, rate of change, y-intercept, ordered pair, slope, linear, line, equation, pattern of association, independent variable, dependent variable

Additional Resources:

- Article: 11 activities that make practicing scatter plot graphs rock
- Activity: Patterns of association—quality of English spoken by people who speak Spanish in their homes
- Video: Bivariate data and analysis: anthropological studies
- Activity: US Airports, assessment variation
- Video: Bungee jump

Data Analysis, Statistics, and Probability

Investigate patterns of association in bivariate data.

21. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects, using relative frequencies calculated for rows or columns to describe possible associations between the two variables.

Guiding Questions with Connections to Mathematical Practices:

How can two-way tables for bivariate data be constructed?

M.P.4. Model with mathematics. Organize the data in a two-way table so that each cell in the table represents the frequency count or a proportion of the two categories. For example, in a survey of seventh- and eighth-grade students and their preference for wearing a hat or not, the two rows will be the grades, and the two columns will be "Wearing a Hat" and "Not Wearing a Hat." As such, each cell will show the number of students in the given grade who have the given preference. Additionally, interpret the information provided in a two-way table.

- Ask students to create a survey for their classmates that asks two binary questions, then ask them to create a two-way table from the results of their survey. For example, a survey of 26 students might find the results given.
    - 3 students run cross country and do not play an instrument
    - 2 students run cross country and do play an instrument
    - 13 students do not run cross country and do not play an instrument
    - 8 students do not run cross country and do play an instrument

  One possible two-way table is shown.

	Cross Country	**No Cross Country**	**Total**
Instrument	2	8	10
No Instrument	3	13	16
Total	5	21	26

- Ask students to interpret a given two-way table. For example, give students the completed two-way table showing information about the 80 participants in a 5-kilometer run.

	Student	Non-student	Total
Ran Entire Time	22.5%	20%	42.5%
Ran and Walked	30%	5%	35%
Walked Entire Time	10%	12.5%	22.5%
Total	62.5%	37.5%	100%

Some possible questions to ask students, along with possible responses, are listed.

- ○ What percentage of the participants ran the entire race?

  Of the 80 participants, 42.5% ran the entire race.

- ○ How many participants were students?

  Of the 80 participants, 62.5% were students, which means that there were $0.625 \times 80 = 50$ students.

- ○ What percentage of the participants were non-students who both ran and walked during the race?

  Of the 80 participants, 5% were non-students who both ran and walked during the race.

- ○ What percentage of the students ran the entire race?

  Of the 80 participants, 62.5% were students and 22.5% were students who ran the entire race. The percentage of students who ran the entire race was $\frac{0.225}{0.625} = 0.36$, or 36%.

- Ask students to complete a two-way table that contains partial information. For example, give students the following two-way table showing information about the groups at a restaurant, and ask them to fill in each empty cell.

	With Kids	Without Kids	Total
Ordered Dessert	15		19
No Dessert	22		
Total		25	

Some possible student reasoning statements are shown.

  - Of the groups with kids, 15 ordered desserts and 22 did not. Therefore, there were a total of $15 + 22 = 37$ groups with kids.
  - Of the 19 groups who ordered desert, 15 were with kids, which means that $19 - 15 = 4$ of the groups that ordered dessert were without kids.
  - Of the 25 groups that were without kids, 4 ordered dessert, which means that $25 - 4 = 21$ of the groups without kids did not order dessert.
  - Of the groups that did not order dessert, 22 were with kids and 21 were without kids. Therefore, there were a total of $22 + 21 = 43$ groups that did not order dessert.
  - There were 19 groups who ordered dessert and 43 that did not, which means that a total of $19 + 43 = 62$ groups were at the restaurant.

How is a two-way table using frequencies related to a two-way table using proportions?

M.P.7. Look for and make use of structure. Describe the relationships between the different data representations in two-way tables and determine how to create one type of two-way table given a different type of two-way table. For example, using a two-way table with percentages, determine the frequencies for each part of the table by finding the percentage of the total number of data points for each cell in the table. If 13% of eighth-grade students bring their lunch to school each day, find the frequency of that data point by finding 13% of the total number of eighth-grade students to represent the information in a different way. Additionally, demonstrate that the totals in a two-way table using proportions should add up to 100%.

- Ask students to describe the difference between the information in a two-way table using frequencies and the information in a two-way table using proportions. For example, tell students that an entry in a two-way table contains the number 18. In a two-way table using frequencies, this would indicate that 18 of the data points fell within a certain category. In a two-way table using proportions, this would indicate that 18% of all the data points fell within a certain category.

- Ask students to convert a two-way table using frequencies or proportions into a two-way table that uses the other representation. For example, give students the following two-way table showing information about toys in a store.

	Not Electronic	Uses Batteries	Uses Outlet	Total
Costs $0 to $20	28	11	2	41
Costs $20 to $50	16	23	6	45
Costs $50 to $100	7	14	21	42
Total	51	48	29	128

There is a total of 128 toys represented in the two-way table. The data in the table can be converted from frequencies to proportions by dividing all numbers by 128 and then representing the quotients as percentages, rounding as necessary. Converting the frequency values in the above table to proportional values expressed as percentages rounded to the nearest tenth yields the following table.

	Not Electronic	Uses Batteries	Uses Outlet	Total
Costs $0 to $20	21.9%	8.6%	1.6%	32.0%
Costs $20 to $50	12.5%	18.0%	4.7%	35.2%
Costs $50 to $100	5.5%	10.9%	16.4%	32.8%
Total	39.8%	37.5%	22.7%	100%

Observe that not all rows and columns add up exactly due to rounding.

How can data in a two-way table be analyzed and interpreted to determine any associations between variables?

M.P.2. Reason abstractly and quantitatively. Analyze a two-way table to determine any patterns that occur. For example, a survey of students in an elementary school who walk or ride the bus to school compared to students in a high school who walk or ride the bus to school might have patterns of association that show that high school students are more likely to walk to school than elementary school students (or vice versa). Additionally, calculate relative frequencies from rows or columns to support conclusions about possible associations between variables.

- Ask students to give examples of variables that are associated and to explain how the variables are associated. For example, give students the variables of the color of a cell phone, the year the cell phone was made, and the quality of the cell phone camera. Then, ask students to determine which of the three variables are associated and how they are associated. The color is unlikely to be associated with either of the other two variables. However, the year and the camera quality are likely to be associated because newer phones are more likely to have better cameras. In a two-way table, this association might be indicated by a larger number of newer phones with higher-quality cameras and a larger number of older phones with lower-quality cameras.

- Ask students to determine whether the data in a given two-way table seem to indicate an association between the variables. For example, give students the following two-way table about intersections in a city.

	Traffic Light	**Stop Sign**	**Total**
Street Lamp	142	98	240
No Street Lamp	34	88	122
Total	176	186	362

When looking at intersections with traffic lights, there are many more intersections that have street lamps, but when looking at intersections with stop signs, the numbers with and without street lamps are very close to equal. Similarly, when looking at intersections with street lamps, there are more intersections that have traffic lights, but when looking at intersections without street lamps, there are more intersections with stop signs. Therefore, intersections that have traffic lights are more likely to have street lamps and vice versa. As an additional example, give students the following two-way table of data about ice cream cones ordered from an ice cream shop.

	Chocolate	**Vanilla**	**Total**
Sugar Cone	78	35	113
Waffle Cone	48	22	70
Total	126	57	183

When looking at orders of chocolate ice cream, 78 out of 126 (about 62%) of the orders have sugar cones and 48 out of 126 (about 38%) of the orders have waffle cones. When looking at orders of vanilla ice cream, 35 out of 57 (about 61%) of the orders have sugar cones and 22 out of 57 (about 39%) of the orders have waffle cones. The corresponding percentages are very close, indicating that the type of ice cream ordered is not associated with the type of cone ordered. Similarly, when looking at orders with sugar cones, 78 out of 113 (about 69%) of the orders are for chocolate and 35 out of 113 (about 31%) are for vanilla. When looking at orders with waffle cones, 48 out of 70 (about 69%) of the orders are for chocolate and 22 out of 70 (about 31%) are for vanilla. The fact that these corresponding percentages match further indicates that there is most likely no association between the type of ice cream ordered and the type of cone ordered.

Key Academic Terms:

bivariate, data, categorical, frequency, two-way table, relative frequency, cell, column, row, percent, decimal, association

Additional Resources:

- Activity: Two-way tables—walking and bicycling to work
- Activity: Thinkport | Two-way tables and associations
- Activity: What's your favorite subject?
- Lesson: Two-way table talk

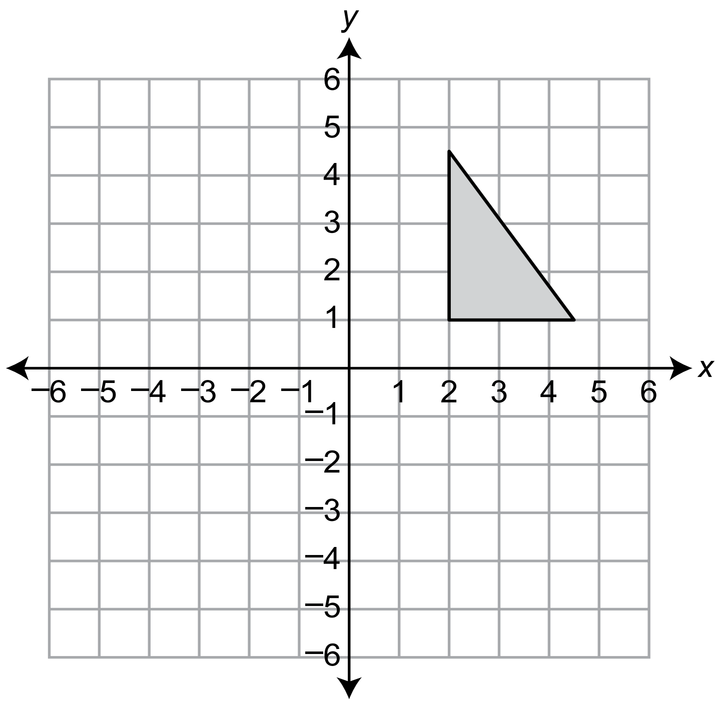
Geometry and Measurement
Understand congruence and similarity using physical models or technology.
22. Verify experimentally the properties of rigid motions (rotations, reflections, and translations): lines are taken to lines, and line segments are taken to line segments of the same length; angles are taken to angles of the same measure; and parallel lines are taken to parallel lines. a. Given a pair of two-dimensional figures, determine if a series of rigid motions maps one figure onto the other, recognizing that if such a sequence exists the figures are congruent; describe the transformation sequence that verifies a congruence relationship.

Guiding Questions with Connections to Mathematical Practices:

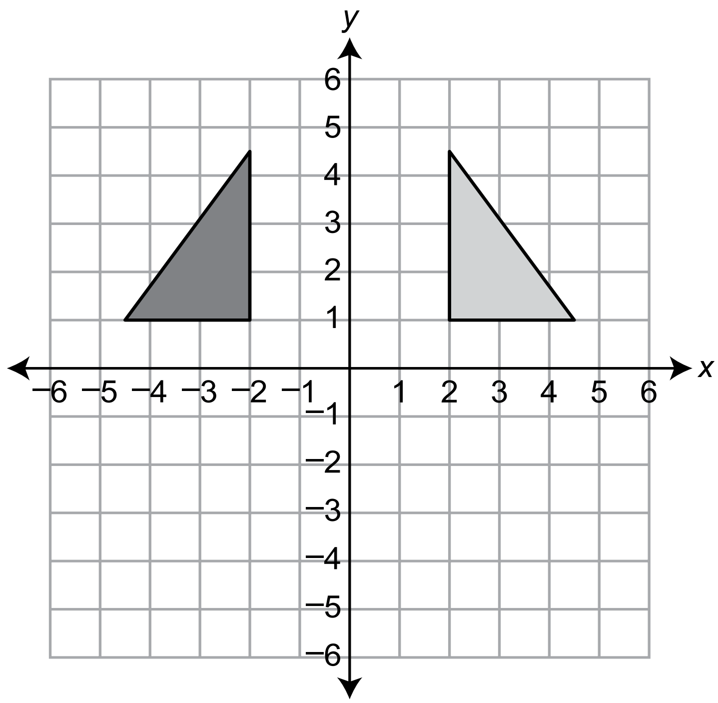
What strategies can be used to illustrate rigid motions?

M.P.5. Use appropriate tools strategically. Demonstrate rigid transformations on a variety of figures using a variety of tools. For example, place a piece of tracing paper over a shape on a piece of paper and trace the shape. Then, slide the tracing paper to a new location to demonstrate a translation. Additionally, use reflective devices, graph paper, or technology to demonstrate a rigid transformation.

- Ask students to demonstrate the results of reflecting an object. Ensure students notice that the rigid transformation of reflection does not change the size or shape of the original figure. For example, ask students to draw a shape in quadrant 1. Have students use graph paper and either paint, a marker that will transfer when wet, or something similar.

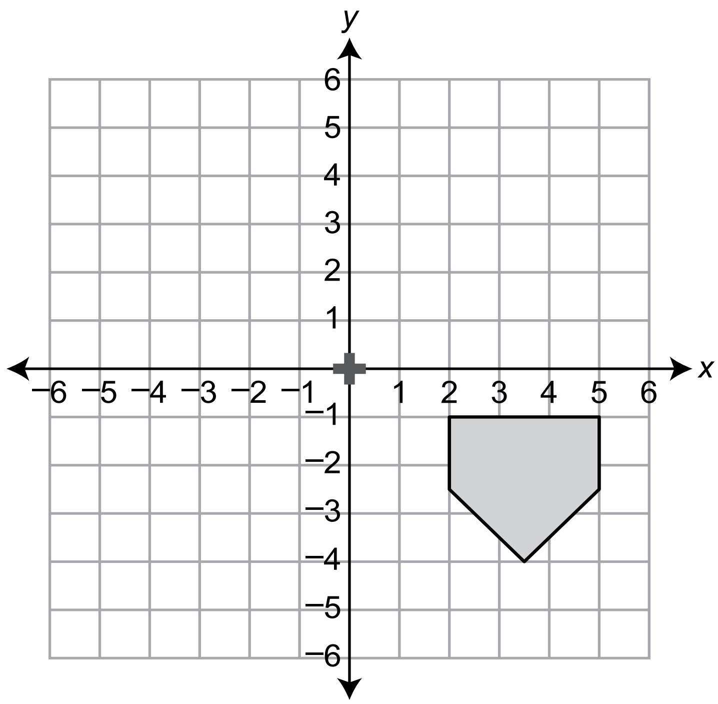


Instruct each student to fold the paper along the *y*-axis and then apply pressure to the fold so that the marker or paint will transfer to the other side of the paper. Then, open the paper again, revealing a "copy" of the original figure now located in quadrant 2 of the graph paper.

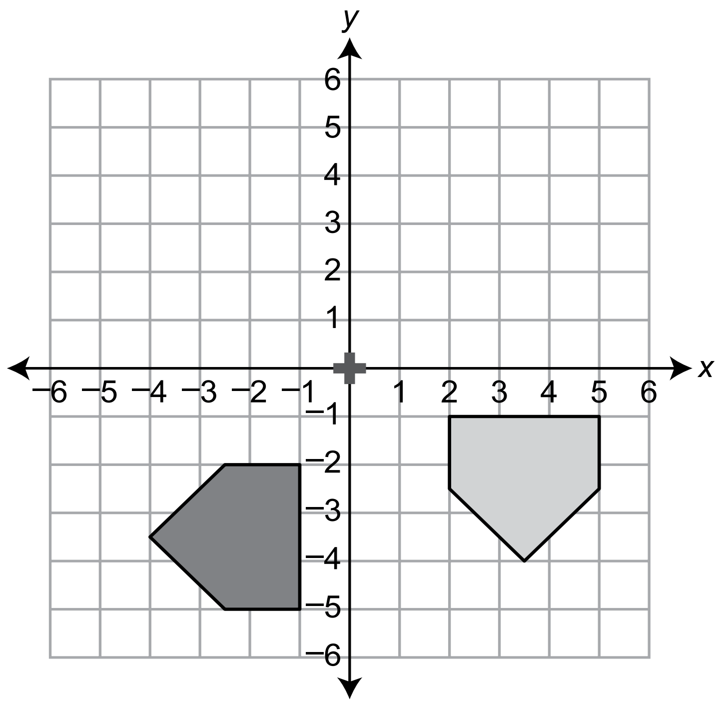


Observe that the new shape is identical to the original shape drawn. This process can also be done using an online graphing tool.

- Ask students to demonstrate the results of rotating an object. Ensure students notice that the rigid transformation of rotation does not change the size or shape of the original figure. For example, use a combination of tracing paper and graph paper to show a rotation. On the graph paper, draw a simple coordinate grid and a shape in any quadrant. Place a piece of tracing paper on top of the coordinate grid and trace the original shape. Students should also trace the axes or mark a small "+" at the origin to help keep track of how far the object has rotated.



Using the origin as the center of rotation, students should place a finger or a pencil on the origin and then slowly rotate the tracing paper. Start by observing the shape when rotated 90 degrees clockwise.

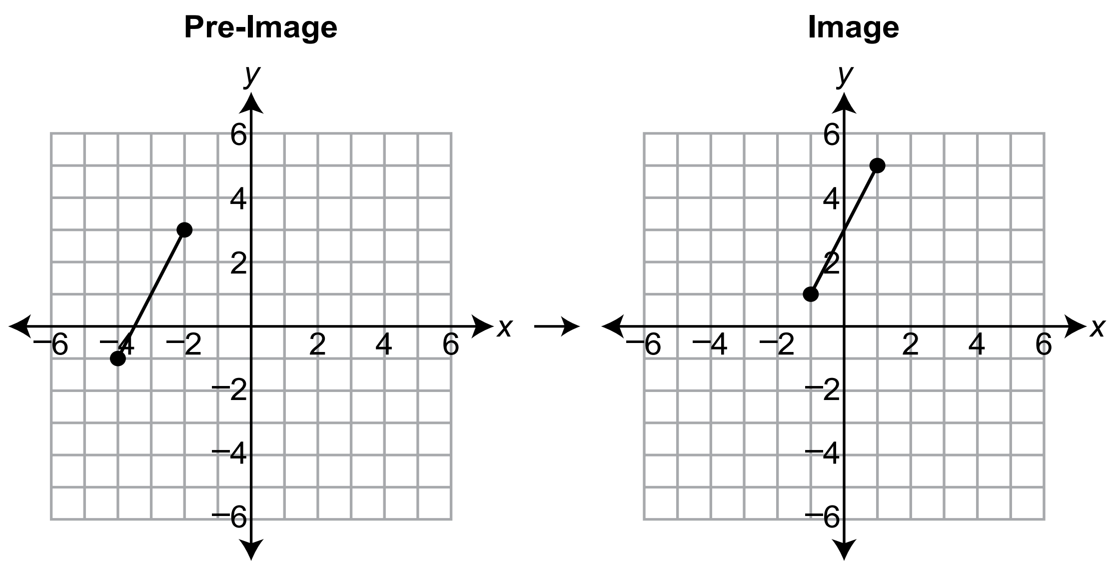


Have students repeat the same process for 180 degrees clockwise and 270 degrees clockwise. Students may also experiment with counterclockwise rotations. Observe that no matter how a shape is rotated, the rotated shapes are identical to the original shape. Have students experiment with the same rotation process using an online graphing tool.

What effects do rotations, reflections, and translations have on lines and line segments?

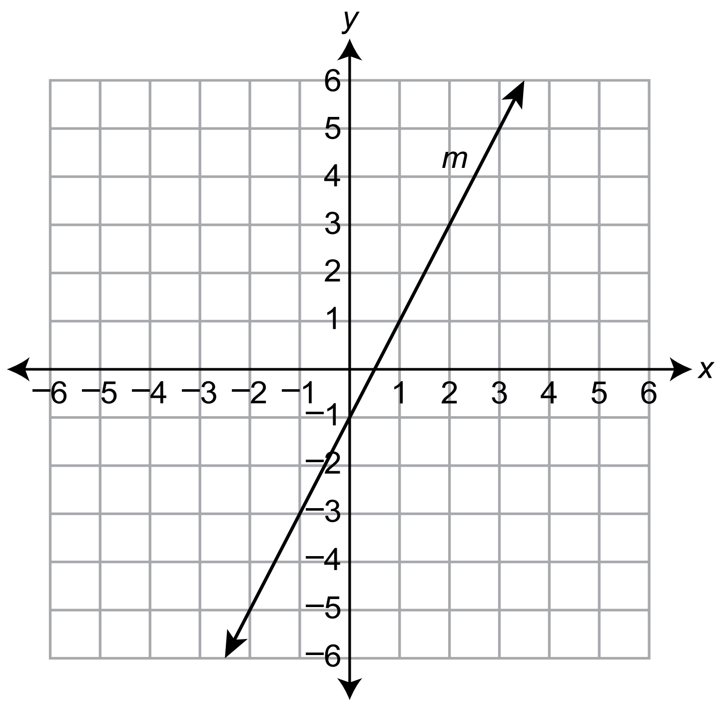
M.P.4. Model with mathematics. Observe rotations, reflections, and translations of lines and line segments to conclude that the size of segments stays the same for these transformations and that both lines and line segments will be oriented differently but will remain lines and line segments of the same length. For example, the line $y = 5x + 1$ rotated 90 degrees clockwise about the origin will remain a line, but the image will have a negative slope and an x-intercept of 1, instead of a positive slope and a y-intercept of 1, like the pre-image. Additionally, a line segment reflected across the x-axis or y-axis will remain a line segment of the same length but will have a new slope as compared to the pre-image.

- Ask students to translate a line segment. Have students plot two points on a coordinate grid and then connect the points to create a line segment, called "the pre-image." Then, ask students to move each of the points 3 units to the right and 2 units upward to create a new line segment, called "the image."

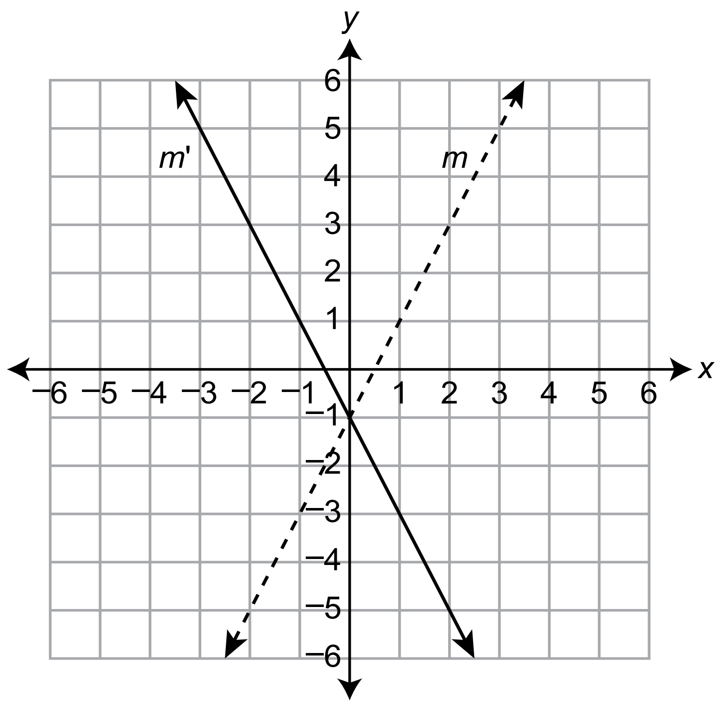


Use a ruler to measure the length of the original line segment (the pre-image). Then, measure the length of the new line segment (the image). Ask students to repeat the process but to draw the line segment in a different quadrant than the original example. Observe that no matter how long their original line segment was, or in which quadrant(s) the line segment was located, the pre-image and image are both line segments of the same length. Have students experiment with translations using an online graphing tool as well.

- Ask students to reflect a line across the y-axis and then to describe the result. For example, students could start by drawing the line $y = 2x - 1$.



Then, reflect the line across the *y*-axis. This could be done in a variety of ways: use a reflective tool to sketch the reflection, draw a dark line that transfers to the other side when the paper is folded on the *y*-axis, use an online graphing tool, or select points on the line and measure their distance from the *y*-axis, then plot a point equidistant from the *y*-axis but on the opposite side.

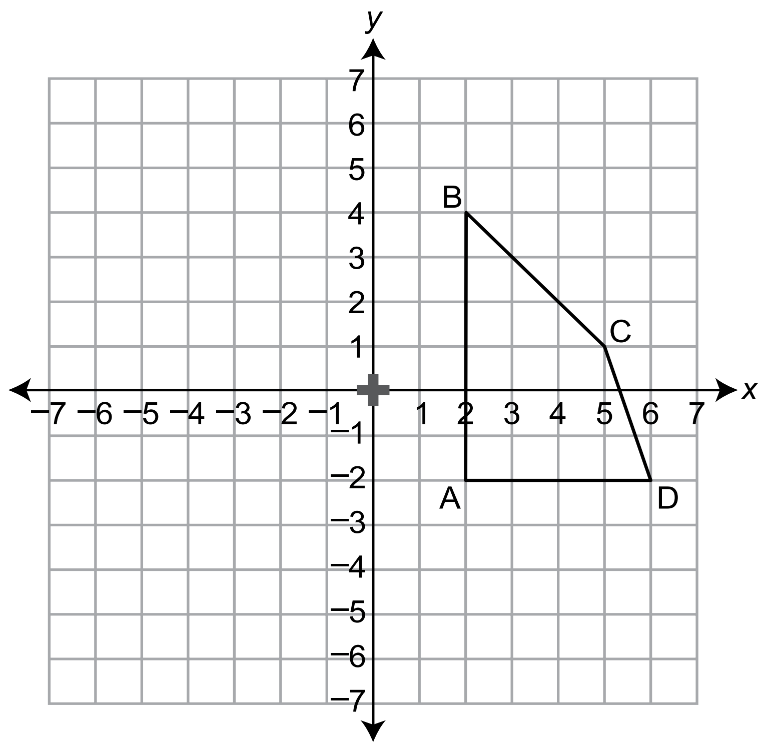


Observe that the image reflected across the *y*-axis has the same *y*-intercept and a new slope. Also observe that the pre-image is a line and that after reflection of the pre-image across the *y*-axis, the image remains a line.

What effects do rotations, reflections, and translations have on angle measures?

M.P.4. Model with mathematics. Observe rotations, reflections, and translations of angles to conclude that the measurement of the angles does not change with rigid motions. For example, a triangle that is reflected across the *y*-axis in a coordinate plane will have the same angle measures as the pre-image. Additionally, a parallelogram translated 4 units to the right and 2 units down will have the same angle measures as the pre-image.

- Ask students to demonstrate that rotating an object preserves the original angle measures of the object. For example, place tracing paper on top of a coordinate grid and draw a shape anywhere on the coordinate grid. Students should also trace the axes or mark a small "+" at the origin to help keep track of how far the object has rotated.

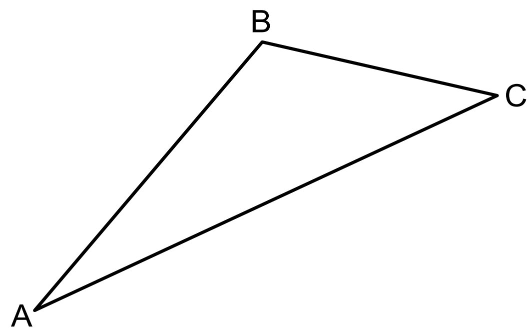


Before rotating, use a protractor to measure all angles of the original shape and record the results in a table. Using the origin as the center of rotation, place a finger or a pencil on the origin and then slowly rotate the tracing paper. First, have students rotate the shape 90 degrees counterclockwise. After this rotation, use a protractor to measure each angle and record the results in the table. Then, repeat this process for rotations 180 degrees counterclockwise, 270 degrees counterclockwise, and 90 degrees clockwise. Students should also use centers of rotation in locations other than the origin, noticing that using a center of rotation at a location other than the origin preserves angle measure as well.

Angle Measures

	Original Angle	Rotate 90° Counterclockwise	Rotate 180° Counterclockwise	Rotate 270° Counterclockwise	Rotate 90° Clockwise
A	90°	90°	90°	90°	90°
B	45°	45°	45°	45°	45°
C	153°	153°	153°	153°	153°
D	72°	72°	72°	72°	72°

- Ask students to demonstrate that translating an object preserves the original angle measures of the object. For example, draw a triangle on a blank sheet of paper.



Use a protractor to measure each angle and record the results in a table, as shown. Then, cut out the triangle and place it anywhere on top of a coordinate grid. Translate the triangle 2 units to the right and 3 units up. Again, use a protractor to measure each angle of the translated triangle and record the results in the table. Then, translate the triangle 6 units to the left and 2 units down and use a protractor to measure and record each angle. Observe that each translation of the object results in angle measures that are the same as those in the original object.

Angle Measures

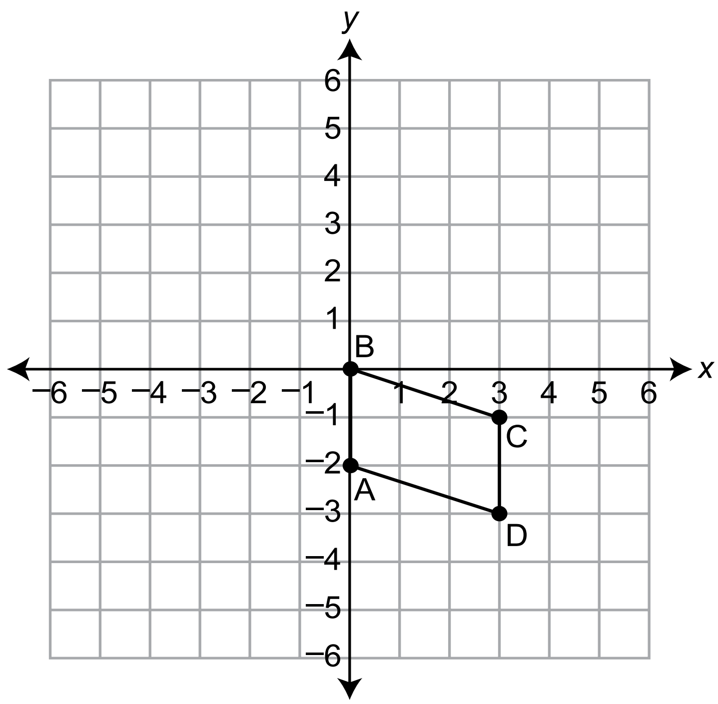
	Original Angle	Translate 2 units right and 3 units up	Translate 6 units left and 2 units down
A	25°	25°	25°
B	116°	116°	116°
C	39°	39°	39°

Repeat the process with shapes of various side lengths, such as squares, rectangles, pentagons, etc. Students should observe that no matter how many sides a shape has, the angle measures stay the same upon translation of the object. Students can also use an online graphing tool to observe that angle measures stay the same when translating objects.

What effects do rotations, reflections, and translations have on parallel lines?

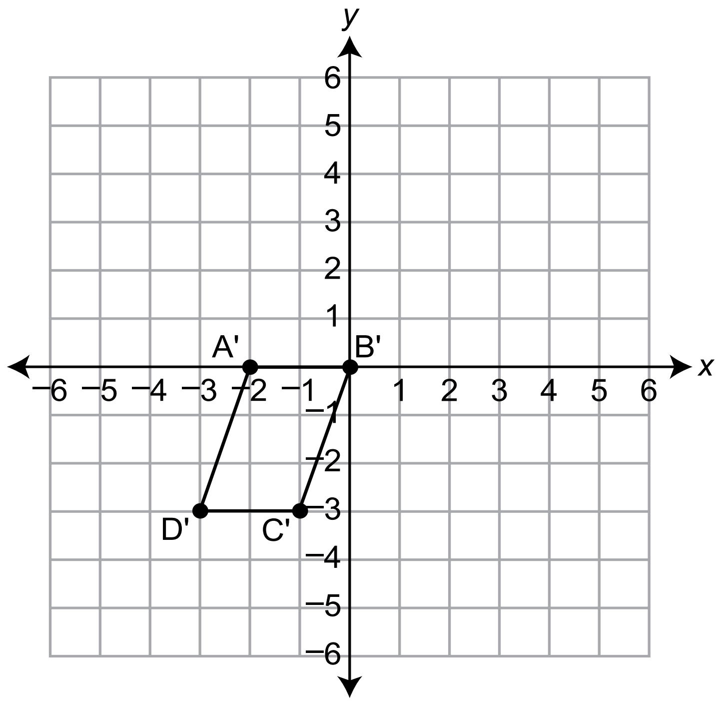
M.P.4. Model with mathematics. Observe rotations, reflections, and translations of parallel lines to conclude that the lines remain parallel with rigid motions. For example, parallel lines that are translated 3 units to the left and then 2 units down on a coordinate plane remain parallel lines with the same space between them. Additionally, parallel lines that are reflected across the x-axis result in two lines with a different slope and y-intercept than the original lines; however, the reflected lines have the same slope as one another, and therefore the reflected lines remain parallel.

- Ask students to demonstrate the result of rotating a parallelogram. For example, draw a parallelogram on a coordinate grid and calculate the slope of each side of the parallelogram.



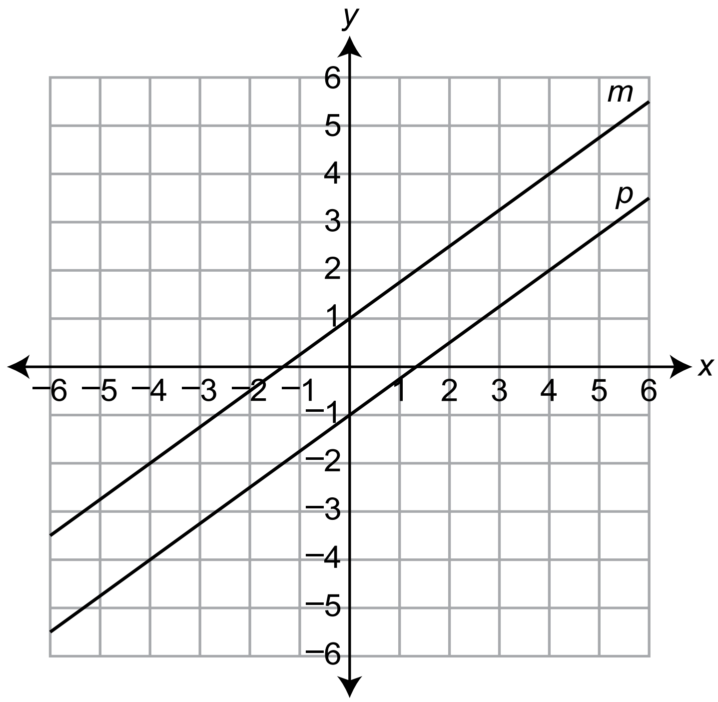
In the figure above, sides AB and CD both have slopes that are undefined. Sides BC and AD both have a slope of $-\frac{1}{3}$. Since the slopes are the same, AB is parallel to CD and BC is parallel to AD.

Now, rotate the parallelogram 90 degrees clockwise and calculate the slope of each side of the rotated figure.



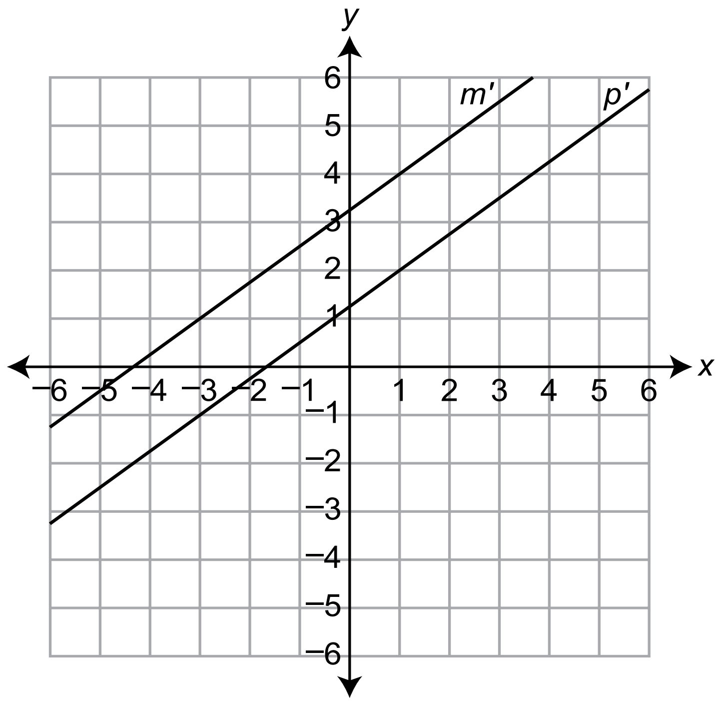
In the rotated figure, the slopes of A'B' and C'D' are both 0 and the slopes of B'C' and A'D' are both 3. Since the slopes of opposite sides are the same, A'B' is parallel to C'D', and B'C' is parallel to A'D'. Observe that AB and CD were parallel in the original figure, and the two sides remain parallel in the rotated figure. Likewise, BC and AD were parallel in the original figure and remain parallel in the rotated figure.

- Ask students to demonstrate the result of translating two parallel lines. For example, draw two parallel lines on a coordinate grid, as shown.



Lines m and p both have a slope of $\frac{3}{4}$. The vertical distance between lines m and p can also be measured at any x-coordinate by either using a ruler or by visually observing using the coordinate grid. Line m is located 2 units above line p.

Ask students to translate both lines 1 unit to the right and 3 units up on the coordinate plane. Then, name the translated lines m' and p'.

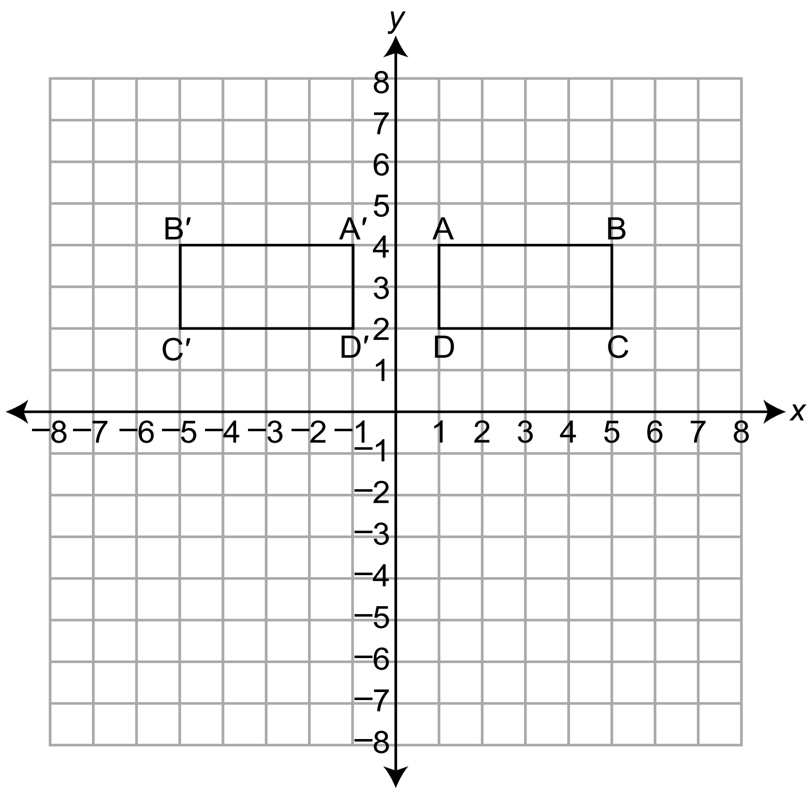


Observe that the slope of each translated line is still $\frac{3}{4}$, and since the slopes of the translated lines are equal, the lines remain parallel. Additionally, line m' remains 2 units above line p', also indicating that the translated lines remain parallel.

How can it be determined that a two-dimensional figure is congruent to another two-dimensional figure?

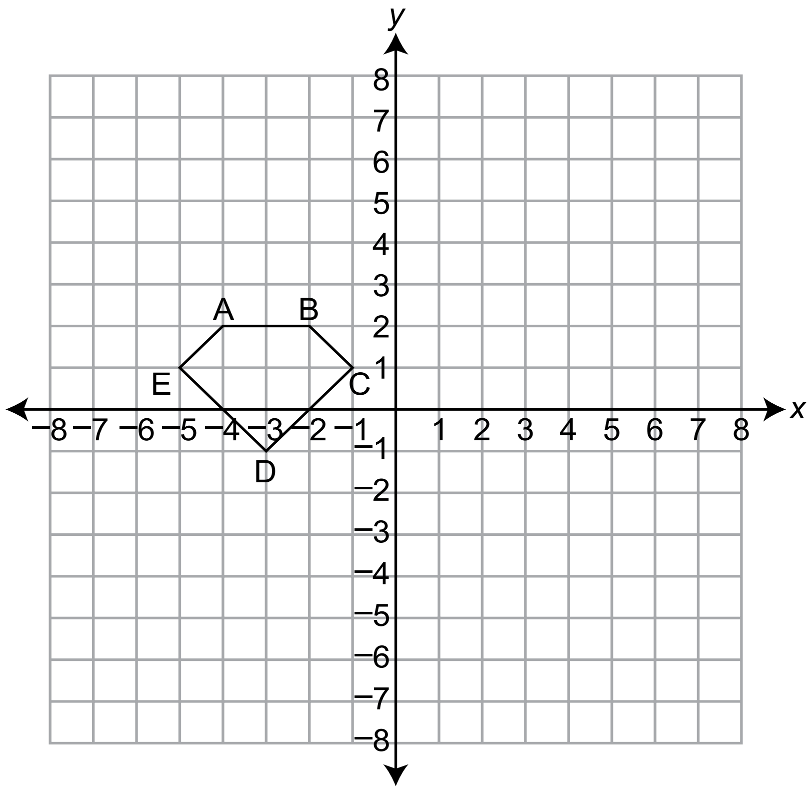
M.P.6. Attend to precision. Show that one figure is congruent to another using rigid motions. For example, demonstrate that two figures are congruent by showing that one figure is the pre-image and one is the same figure after being reflected across a line and then rotated 90° clockwise. Additionally, show that a translation of a figure in any direction results in a figure with the same size and shape.

- Ask students to show that the reflection of a rectangle results in a rectangle that is congruent to the original rectangle. Draw rectangle ABCD, the pre-image, on a coordinate grid, then find and record the angle measures and side lengths. Next, draw rectangle A'B'C'D', which is the image of rectangle ABCD after a reflection across the *y*-axis, as shown.



Record the angle measures and side lengths of rectangle A'B'C'D' and determine that all angle measures and side lengths of the image are the same as the angle measures and side lengths of the pre-image. Next, create other figures and reflect them across the *x*- or *y*-axis. Verify that all side lengths and angle measures remain the same from pre-image to image with any reflection. This activity can also be done using rotations of 90°, 180°, 270°, or 360° clockwise or counterclockwise about the origin.

- Ask students to show that the translation of a pentagon in any direction results in a pentagon that is the same size and shape as the original pentagon. Draw pentagon ABCDE on a coordinate grid, as shown.

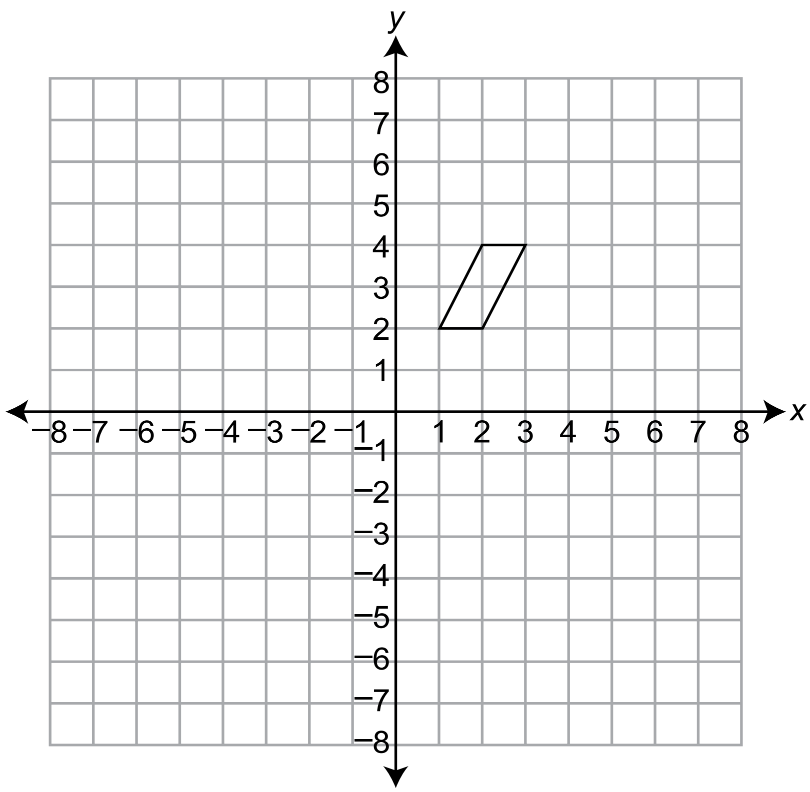


Then, place tracing paper on top of the coordinate grid and trace the pentagon. Translate pentagon ABCDE to the right 5 units and down 3 units by moving the tracing paper to the right and down on the coordinate grid. Discuss whether the size or shape of the pentagon changed from the pre-image to the image. Next, use the tracing paper to model more translations of pentagon ABCDE in any direction. Discuss whether the size or shape of the pentagon changes during a translation. Then, draw other figures to translate and verify that the size and shape of the figures remain the same from pre-image to image.

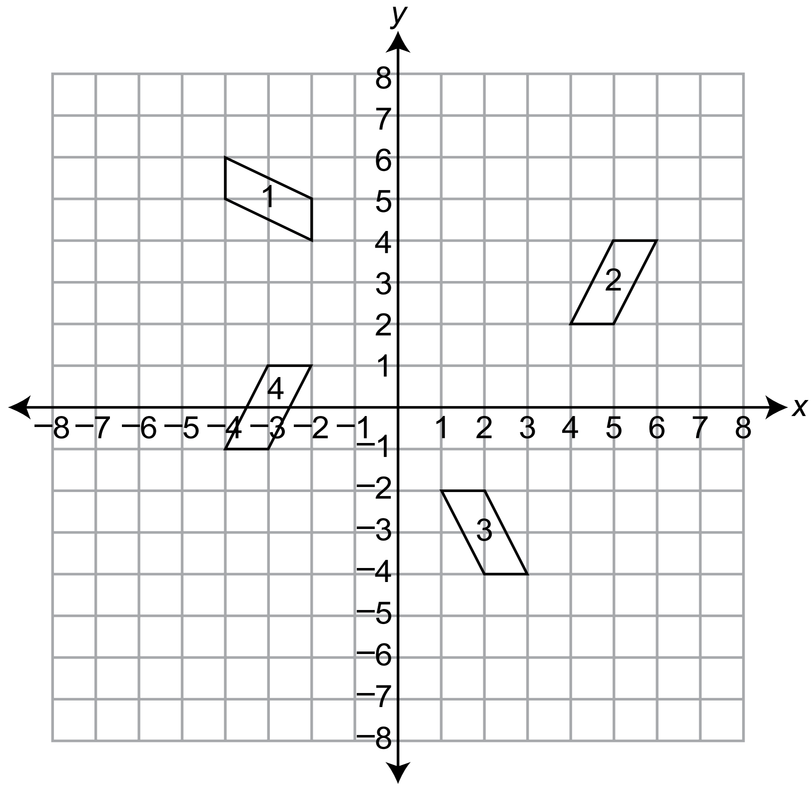
How can a strategy be determined to describe a sequence of transformations that shows congruency between figures?

M.P.1. Make sense of problems and persevere in solving them. Analyze two figures to determine a sequence of transformations that show congruency. For example, when using a mirror, geometry software, or paper folding, if a figure and its pre-image are determined to be mirror images of each other, reflection must be included in the sequence of transformations because mirror images maintain congruent angle measures and side lengths. Additionally, if an image maintains the same size, shape, and orientation as the pre-image, a translation or sequence of translations maps the pre-image onto the image.

- Ask students to determine which figures could be the images of a given pre-image after undergoing transformations involving only a translation or series of translations. Give students the pre-image of a figure graphed on a coordinate grid.



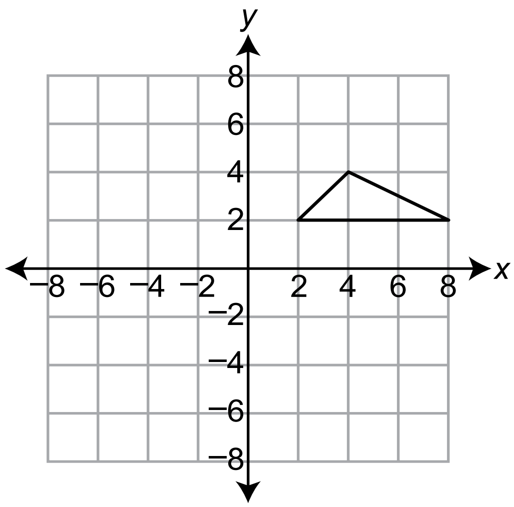
Then, give students several other figures graphed on a coordinate grid.



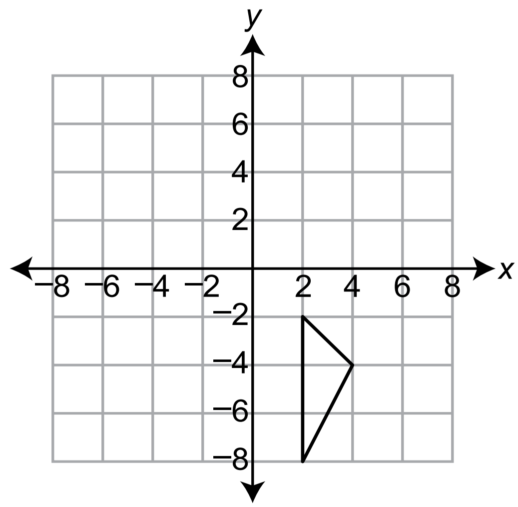
Ask students to determine which figures could be the images of the given pre-image after transformations involving only a translation or series of translations. Then, determine the transformations that map the pre-image onto the identified images. Determine that figure 2 is the image after a translation to the right 3 units and that figure 4 is the image after a translation to the left 5 units and down 3 units. Discuss the similarities between figures 2 and 4. Also, ask students to discuss why figures 1 and 3 are not possible images of the given pre-image after undergoing transformations involving only translations.

- Ask students to determine characteristics of figures that have been rotated about a fixed point. Begin by graphing a triangle on a coordinate grid. Then, rotate the triangle 90° clockwise about the origin and graph the image on another coordinate grid. Repeat this process two more times. Students should also find that after rotating the triangle 360° about the origin, it will map onto the original triangle.

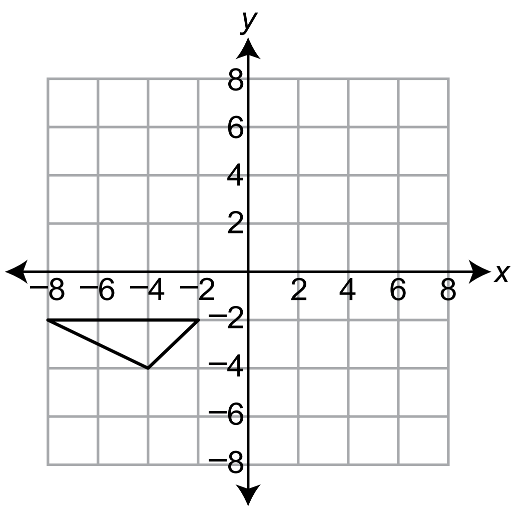
Original Triangle



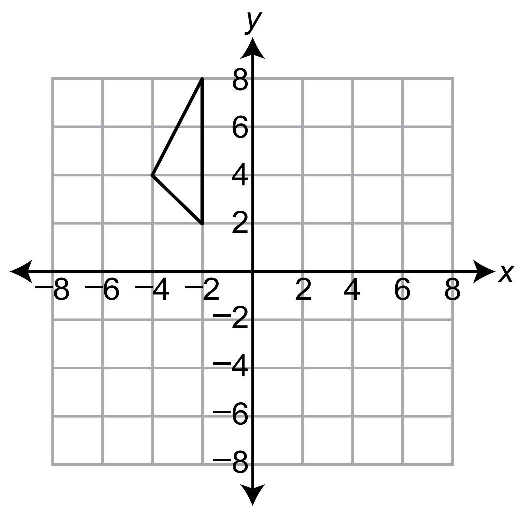
Rotated 90° Clockwise



Rotated 180° Clockwise

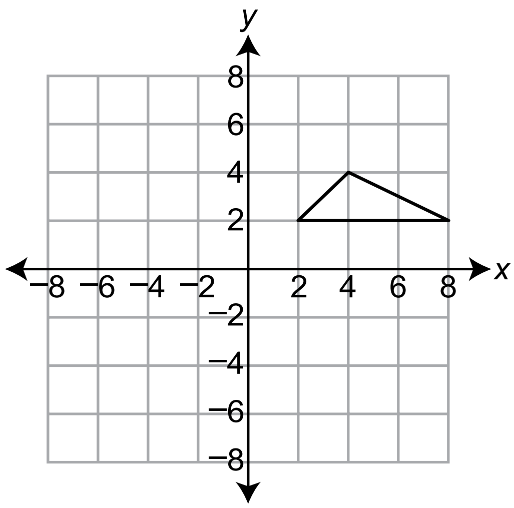


Rotated 270° Clockwise

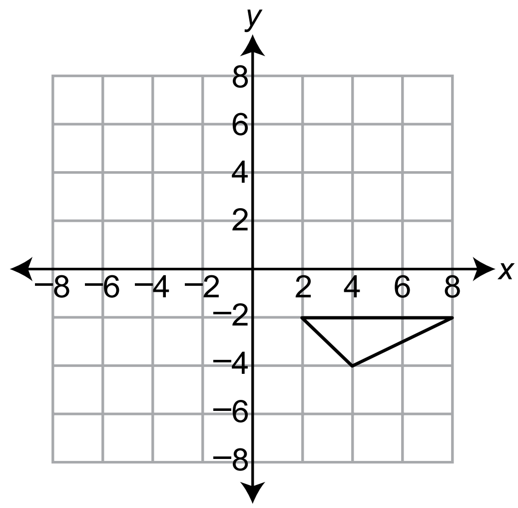


Discuss the characteristics that remain the same and the characteristics that change when the triangle is rotated. Next, revisit the original triangle graphed on the coordinate grid and ask students to graph a reflection of the triangle across the *x*-axis, as shown.

Original Triangle



Reflected Across *x*-axis



Discuss the similarities and differences between the rotations and reflections of the triangle as well as the characteristics of the triangles, such as side lengths, angle measures, orientation, and position.

Key Academic Terms:

rotation, reflection, translation, transformation, rigid motion, image, pre-image, line, line segment, center of rotation, line of reflection, clockwise, counterclockwise, corresponding, map, angle, parallel, sequence, congruent, mirror image, *x*-axis, *y*-axis, ordered pair

Additional Resources:

- Activity: Rotational symmetry
- Activity: Congruent rectangles
- Activity: Reflections, rotations, and translations
- Lesson: Introduction to congruence and similarity through transformations
- Activity: Cutting a rectangle into two congruent triangles
- Lesson: Translations
- Video: Monkey around
- Video: Congruent vs similar triangles
- Lesson: Hands-on exploring the movement of reflections in the plane
- Activity: Transformation Golf: Rigid motion
- Activities: Transformation Worksheets

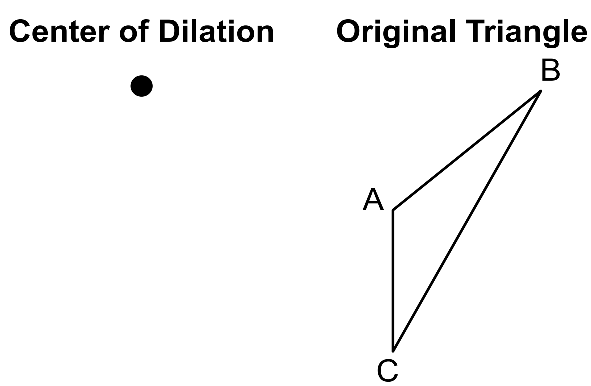
Geometry and Measurement
Understand congruence and similarity using physical models or technology.
23. Use coordinates to describe the effect of transformations (dilations, translations, rotations, and reflections) on two-dimensional figures.

Guiding Questions with Connections to Mathematical Practices:

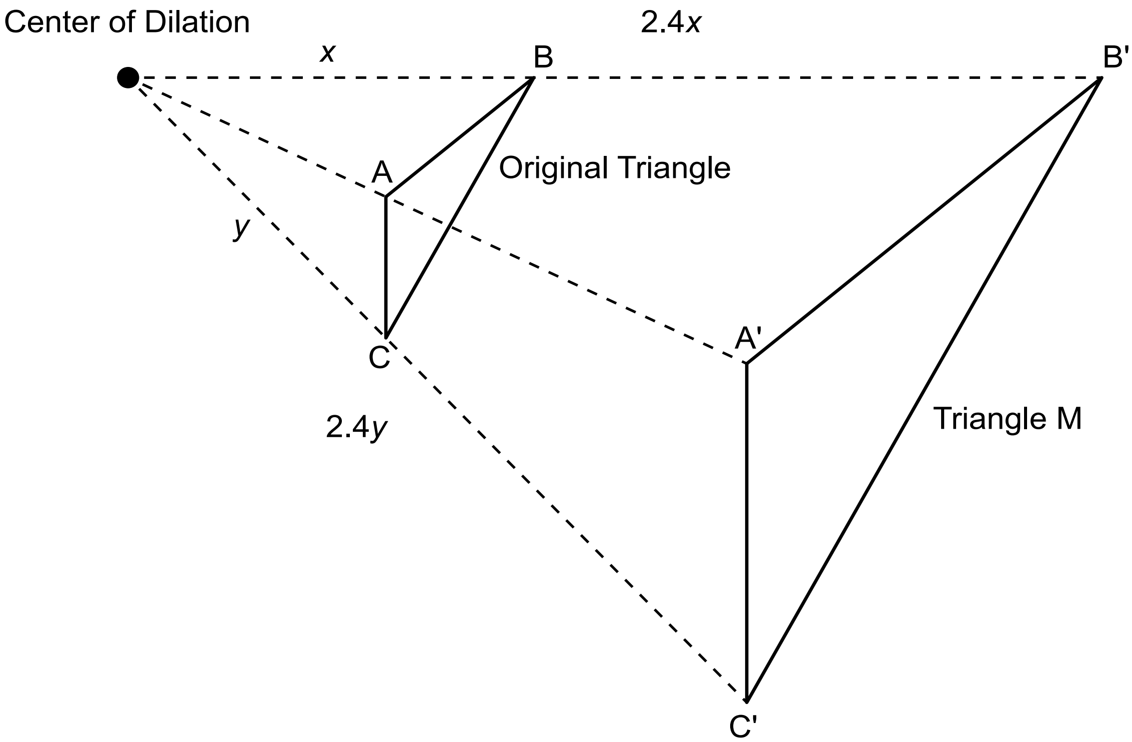
How are dilations different from other types of transformations?

M.P.2. Reason abstractly and quantitatively. Describe the transformation of dilation as making an image that is similar to the original image (pre-image); specifically, dilation creates an image that is the same shape but a different size than the original. Dilations change the size of the pre-image, whereas translations, rotations, and reflections do not change the size of the pre-image. For example, a rectangle that is dilated by a factor of 3 will still be a rectangle with four right angles, but the side lengths for the image will be 3 times the length of the corresponding sides of the pre-image. Additionally, a triangle that is dilated by a factor of $\frac{1}{2}$ will still be a triangle with three angles that are equivalent to the angles in the original figure, but the side lengths will be $\frac{1}{2}$ the length of the corresponding sides of the pre-image.

- Ask students to determine the scale factor of a dilation given the original figure and the figure after it has been dilated. For example, give students a triangle and have them measure its side lengths using rulers.



Then, show students how to use the center of dilation to draw a similar triangle with different side lengths than the original triangle but congruent corresponding angle measures.



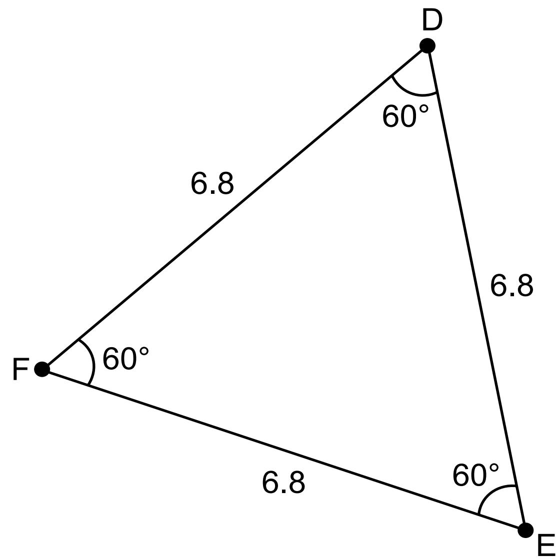
Measure the side lengths of the dilated triangle, triangle M, and record the results in the table. Then, find the scale factor of each dilation using the side lengths.

	Side AB	Side BC	Side AC	Scale Factor
Original Triangle	2.6 cm	4.1 cm	2.0 cm	---
Triangle M	6.2 cm	9.8 cm	4.8 cm	2.4

Since $\frac{2.6}{6.2} = \frac{4.1}{9.8} = \frac{2.0}{4.8}$, the sides of the triangles are proportional; therefore, triangle M is similar to the original triangle.

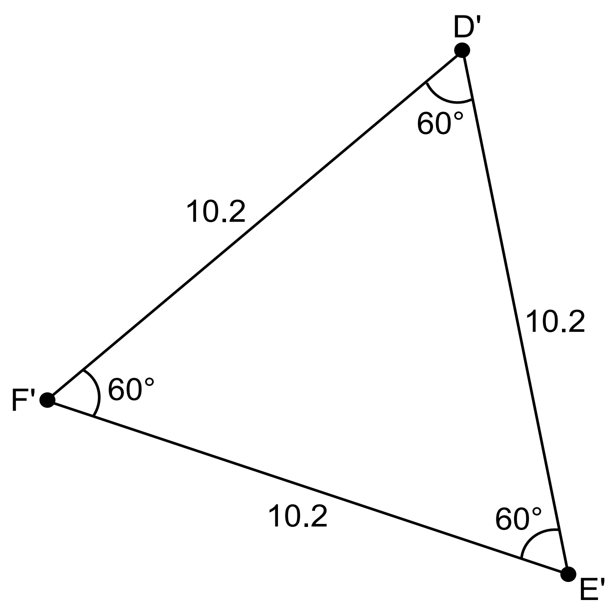
- Ask students to use technology to demonstrate a dilation of a figure. For example, use an online graphing tool to create a figure and label the figure "Pre-image." Then, measure the side lengths and angles of the pre-image and add measurements to the figure, as shown.

Pre-image



After adding the measurements, dilate the figure about its center by a scale factor of 1.5 and find the side lengths and angle measures of the dilated figure. Observe that all the angle measures of the dilated image are the same as the angle measures of the pre-image. Also find that the side lengths of the pre-image can all be multiplied by the same number (the scale factor) to find the corresponding side lengths of the image. Then, label the dilated image with the scale factor, as shown.

Dilation with Scale Factor 1.5

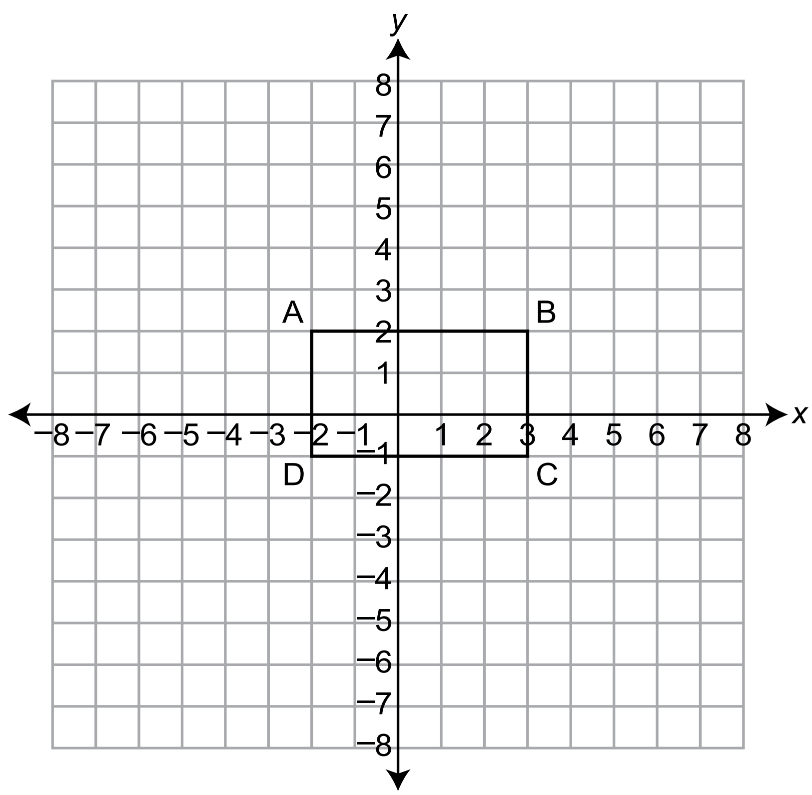


How can the coordinates of a figure be determined when a dilation is performed on the pre-image of the figure?

M.P.7. Look for and make use of structure. Identify the coordinates of the pre-image and use the description of the dilation to find the coordinates of the image. For example, to find the coordinates of an image of a square with the coordinates (2, 2), (6, 2), (6, 6), and (2, 6) under a dilation of $\frac{1}{2}$ with a center of dilation at the origin, start with the coordinates of the pre-image square and multiply each coordinate by $\frac{1}{2}$. The image of the square will have the coordinates (1, 1), (3, 1), (3, 3), and (1, 3). Additionally, to determine the coordinates of an image of a triangle with pre-image coordinates (1, 1), (3, 4), and (4, 1) under a dilation of 3 with a center of dilation at the origin, multiply each coordinate by 3. The image of the triangle will have the coordinates (3, 3), (9, 12), and (12, 3).

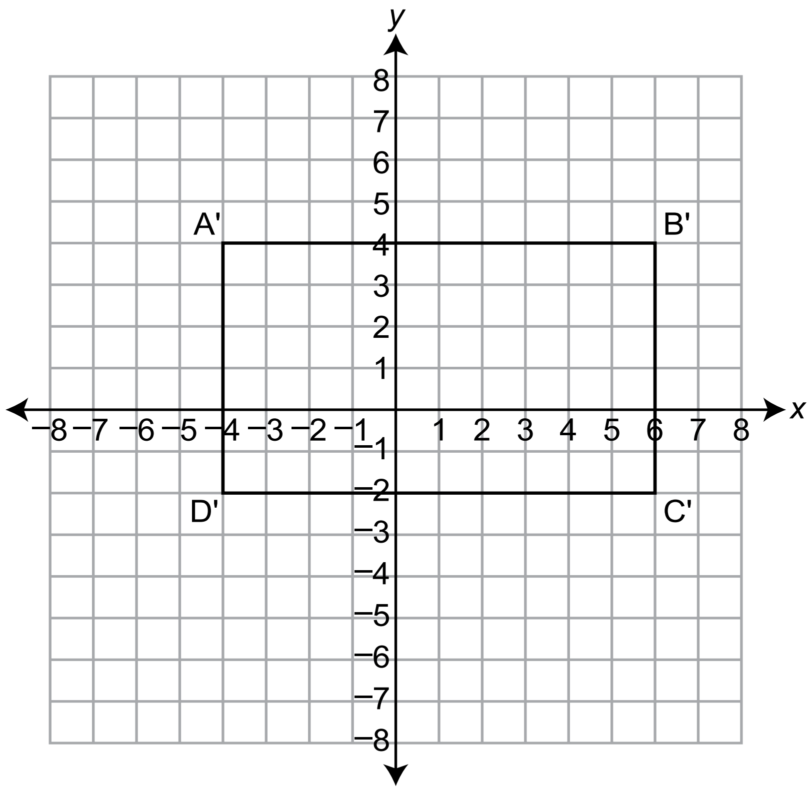
- Ask students to find the coordinates of the vertices of an image that has been dilated on a coordinate grid. For example, give students the rectangle ABCD, the pre-image, graphed on a coordinate grid.

Pre-image



Then, dilate rectangle ABCD by a scale factor of 2 with the center of dilation at the origin. Multiply each x- and y-coordinate by 2 for each vertex of rectangle ABCD and then use the coordinates of the vertices of the image to graph rectangle A'B'C'D' on the coordinate grid.

Image



Verify that the side lengths of the image are twice the length of the side lengths of the pre-image and that the angles of the pre-image and the image are all right angles. Discuss with students that the strategy of multiplying the coordinates by the scale factor only works when the center of dilation is at the origin and that a different strategy will be used when the center of dilation is not at the origin.

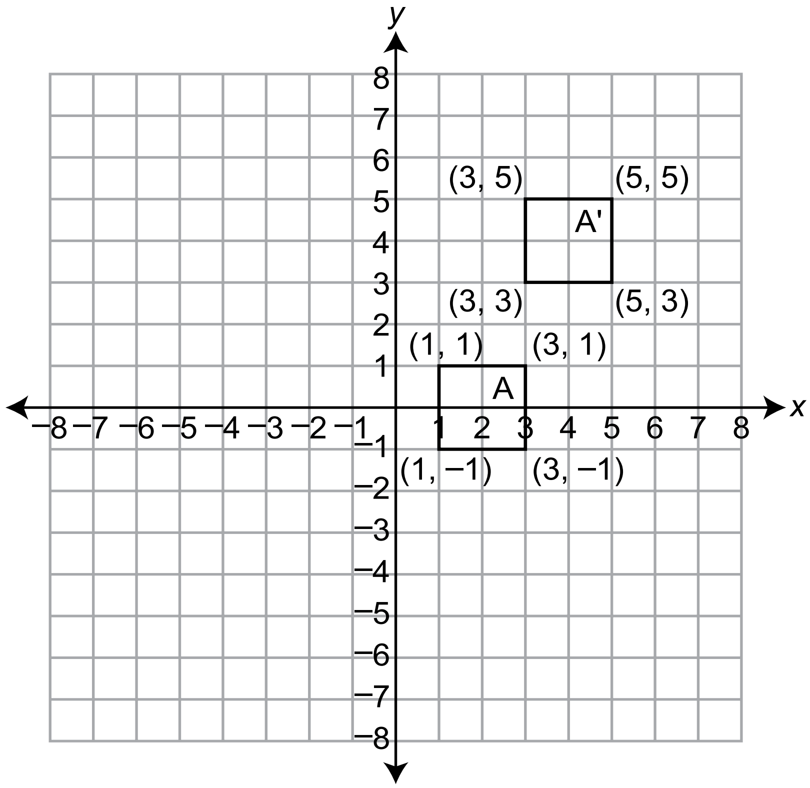
- Ask students to use a table of values to find the coordinates of the vertices of an image that has been dilated with a center of dilation at the origin. For example, give students a table with the coordinates of the vertices of the pre-image of a triangle. Then, find the coordinates of the vertices of the image of the triangle after undergoing the designated dilation listed in the table. Multiply each x- and y-coordinate by the scale factor to determine the coordinates of the dilated image and record the findings in the table.

	Vertex A	**Vertex B**	**Vertex C**
Pre-Image: Triangle	(−4, −1)	(−1, 2)	(2, −3)
dilate by a scale factor of 3 centered at the origin	(−12, −3)	(−3, 6)	(6, −9)
dilate by a scale factor of $\frac{1}{2}$ centered at the origin	$(-2, -\frac{1}{2})$	$(-\frac{1}{2}, 1)$	$(1, -\frac{3}{2})$

How can the coordinates of a figure be determined when a rigid motion is performed on the pre-image of the figure?

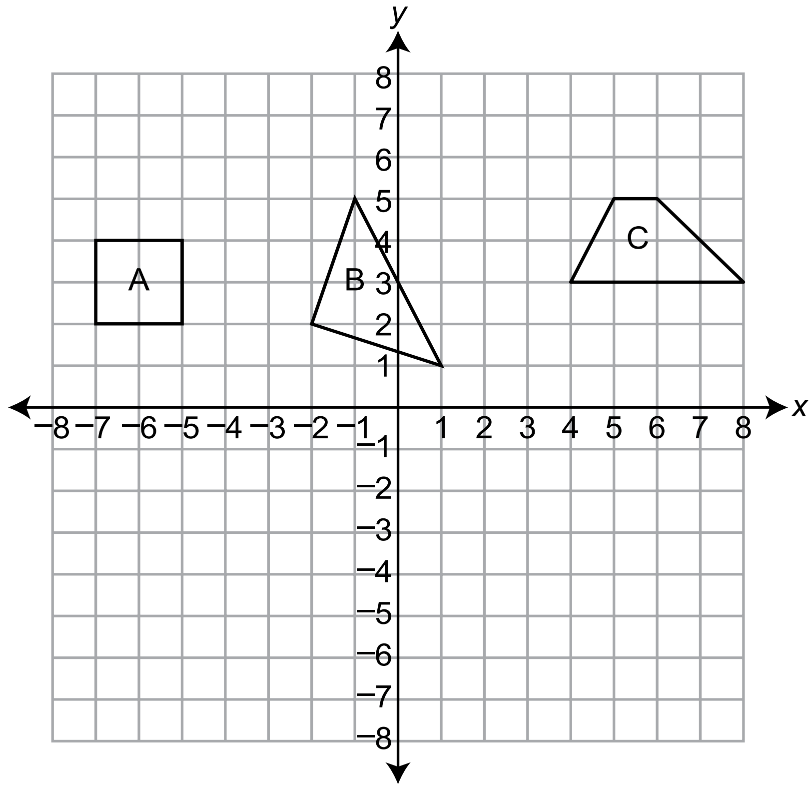
M.P.7. Look for and make use of structure. Identify the coordinates of the pre-image and use the description of the translations, rotations, and/or reflections to find the coordinates of the new figure. For example, to find the new coordinates of a triangle that is translated 4 units to the left, subtract 4 from each x-coordinate. Therefore, a triangle with vertices at (1, 2), (3, 1), and (4, 4) translated 4 units to the left will have the new coordinates (−3, 2), (−1, 1), and (0, 4). Additionally, to find the new coordinates of a rectangle that is reflected across the y-axis, observe that the value of the x-coordinate of each vertex will be the opposite of the value of the x-coordinate of the vertex of the original rectangle and the y-coordinate will remain the same.

- Ask students to find patterns in the coordinates of the vertices of an image of a square that has been translated on a coordinate grid in several different ways. For example, give students the pre-image, square A, graphed on a coordinate grid and translate it to the right 2 units and up 4 units to create square A', as shown.



Note that the x-coordinate of each vertex of square A' is 2 more than the x-coordinate of the corresponding vertex of square A. Similarly, the y-coordinate of each vertex of square A' is 4 more than the y-coordinate of the corresponding vertex of square A. Then, ask students to translate the original square in other ways, using different amounts of units in a variety of directions and discuss what happens to the coordinates of the vertices of the square each time it is translated. As students explore translating square A in a variety of ways, discuss noticeable patterns about the changes in the coordinates during a translation. Find that translations to the left result in the subtraction of the number of units from the x-coordinate of each vertex, while translations down result in the subtraction of the number of units from the y-coordinate of each vertex. Also, translations right result in the addition of the number of units to the x-coordinate of each vertex, while translations up result in the addition of the number of units to the y-coordinate of each vertex.

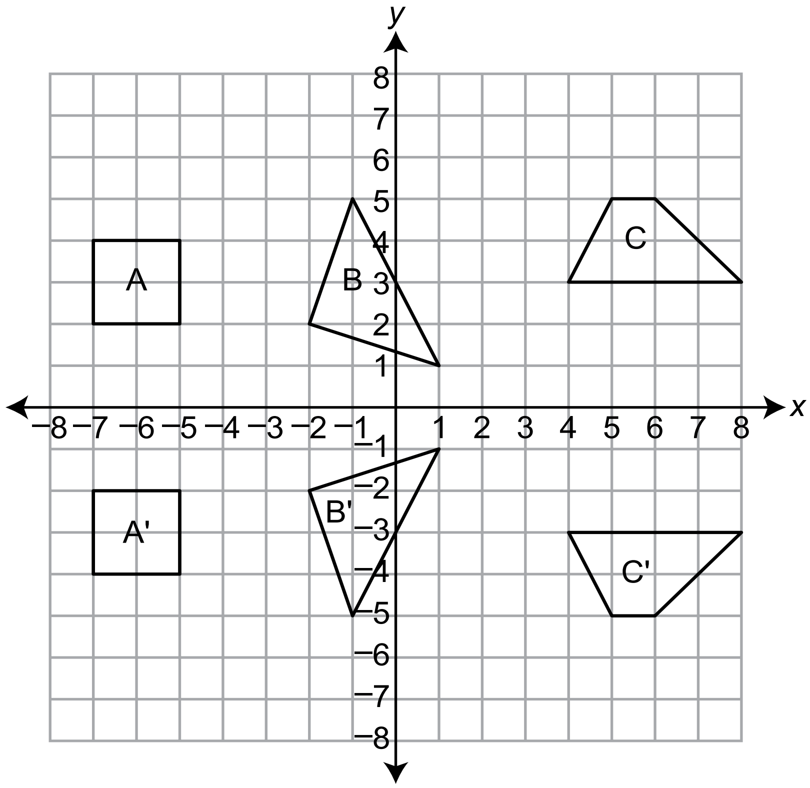
- Ask students to find patterns in the coordinates of the vertices of images of figures that have been reflected across an axis on a coordinate grid. For example, give students figures A, B, and C on a coordinate grid.



Then, ask students to record the coordinates of the vertices of the three figures in a table.

	Coordinates of Original Vertices	**Coordinates of Vertices After Reflection Across *x*-axis**
Figure A	(−7, 2)	
	(−5, 2)	
	(−7, 4)	
	(−5, 4)	
Figure B	(−2, 2)	
	(−1, 5)	
	(1, 1)	
Figure C	(4, 3)	
	(8, 3)	
	(5, 5)	
	(6, 5)	

- Next, reflect each figure across the x-axis.



Then, record the coordinates of the vertices of the reflected figures, A', B', and C', in the tables.

	Coordinates of Original Vertices	Coordinates of Vertices After Reflection Across x-axis
Figure A	(−7, 2)	(−7, −2)
	(−5, 2)	(−5, −2)
	(−7, 4)	(−7, −4)
	(−5, 4)	(−5, −4)
Figure B	(−2, 2)	(−2, −2)
	(−1, 5)	(−1, −5)
	(1, 1)	(1, −1)
Figure C	(4, 3)	(4, −3)
	(8, 3)	(8, −3)
	(5, 5)	(5, −5)
	(6, 5)	(6, −5)

Ask students to discuss what they notice about the x- and y-coordinates of each figure before and after the reflection across the x-axis. Determine that a reflection across the x-axis results in an opposite value for the y-coordinate of each reflected vertex. Repeat this activity with new figures reflected across the y-axis. When students have established rules for coordinates of figures reflected across each axis, practice the reflections using coordinates only. Give students the coordinates of a new figure, triangle ABC, in the table.

	Coordinates of Original Vertices	**Coordinates of Vertices After Reflection Across *x*-axis**	**Coordinates of Vertices After Reflection Across *y*-axis**
Triangle ABC	A(−1, 3)		
	B(4, −2)		
	C(5, 6)		

Then, students should determine the coordinates of triangle ABC after a reflection across the x-axis and after a reflection across the y-axis.

	Coordinates of Original Vertices	**Coordinates of Vertices After Reflection Across *x*-axis**	**Coordinates of Vertices After Reflection Across *y*-axis**
Triangle ABC	A(−1, 3)	(−1, −3)	(1, 3)
	B(4, −2)	(4, 2)	(−4, −2)
	C(5, 6)	(5, −6)	(−5, 6)

Check the answers by graphing the images on a coordinate grid.

Key Academic Terms:

rotation, reflection, translation, transformation, rigid motion, image, pre-image, congruent, similar, preserve, coordinates, dilation, vertex, corresponding, map, proportional, scale up, scale down

Additional Resources:

- Lesson: Math + Arts | Rotation, reflection, and translation in dance
- Lesson: Math + Arts | Symmetry, reflective drawing, and totem poles
- Activity: Reflections, rotations, and translations
- Video: Human tree: dilations
- Video: Rotation and dilation
- Video: Understanding dilations
- Video: Escaramuza: coordinates, reflection, rotation
- Activity: Working with Dilations
- Activity: Dilations Exploration

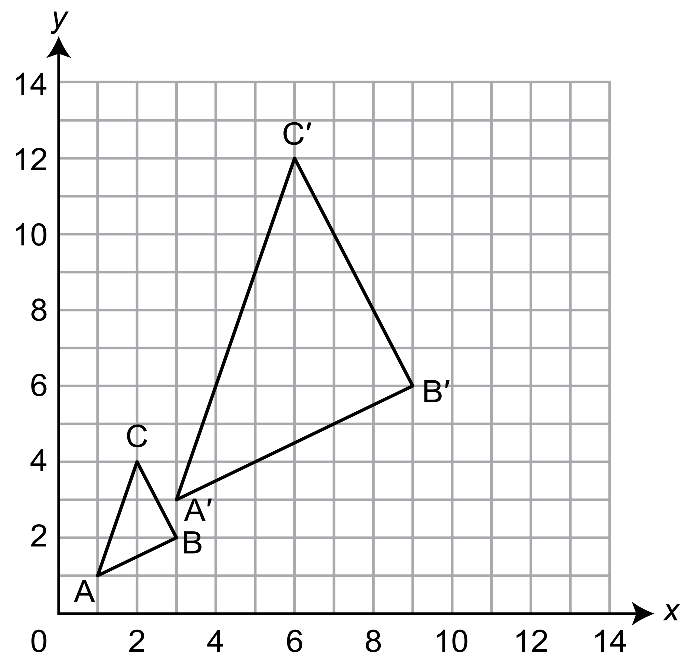
Geometry and Measurement
Understand congruence and similarity using physical models or technology.
24. Given a pair of two-dimensional figures, determine if a series of dilations and rigid motions maps one figure onto the other, recognizing that if such a sequence exists the figures are similar; describe the transformation sequence that exhibits the similarity between them.

Guiding Questions with Connections to Mathematical Practices:

How can it be determined that a two-dimensional figure is similar to another two-dimensional figure?

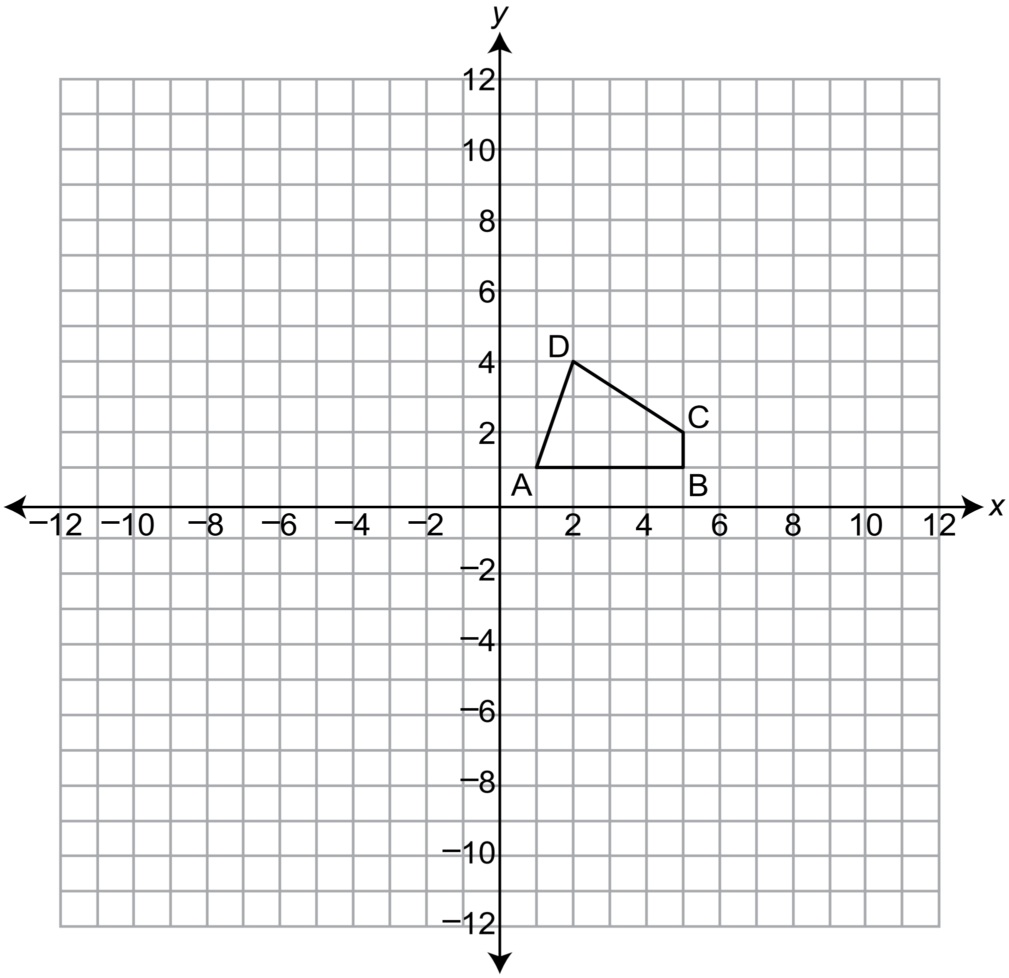
M.P.8. Look for and express regularity in repeated reasoning. Examine the figures' angles and side lengths to determine similarity. For example, a shape that is reflected across a line, rotated 90 degrees counterclockwise, and dilated by a factor of 4 will have equal angle measures to the pre-image and side lengths that are all 4 times longer than the pre-image, making it similar to the pre-image. Additionally, an image translated 6 units to the right and then dilated by a factor of $\frac{1}{2}$ will have equal angle measures to the pre-image and side lengths that are $\frac{1}{2}$ as long as the original, making it similar to the pre-image.

- Ask students to determine similarity when two shapes are given. For example, show students a pre-image, triangle ABC, and the image, triangle A'B'C'.

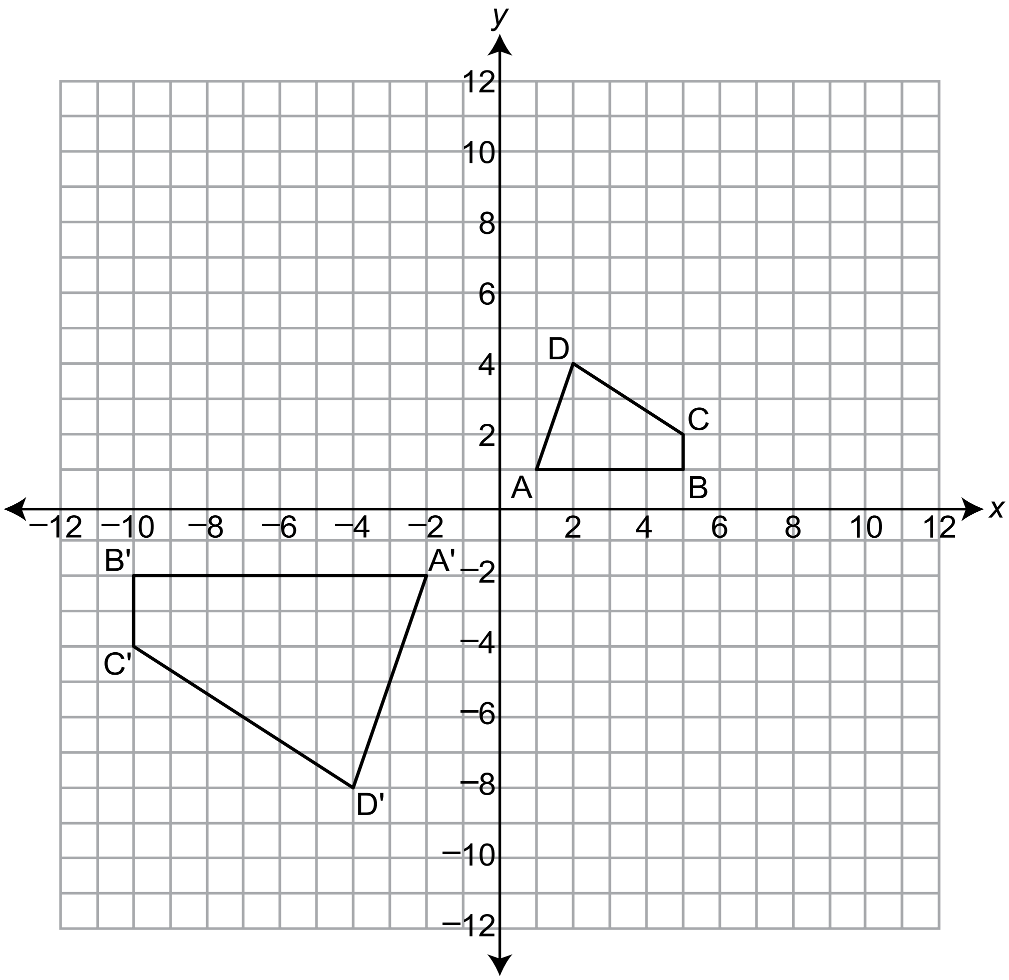


Then, measure the side lengths and angle measures of both triangles. Confirm that the corresponding angles of ABC and A'B'C' are equal and confirm that the side lengths of A'B'C' are three times larger than the side lengths of triangle ABC. Determine that A'B'C' is a dilation of triangle ABC by a scale factor of 3, which makes triangle A'B'C' similar to triangle ABC.

- Ask students to determine whether two shapes are similar when given a shape, the pre-image, and a series of transformations to be performed on the shape to create a new shape. For example, start with a pre-image of a quadrilateral with vertices at A(1, 1), B(5, 1), C(5, 2), and D(2, 4).



Next, rotate the quadrilateral counterclockwise 180 degrees, and then dilate the quadrilateral with a scale factor of 2 with the center of dilation at the origin. Plot the resulting quadrilateral.

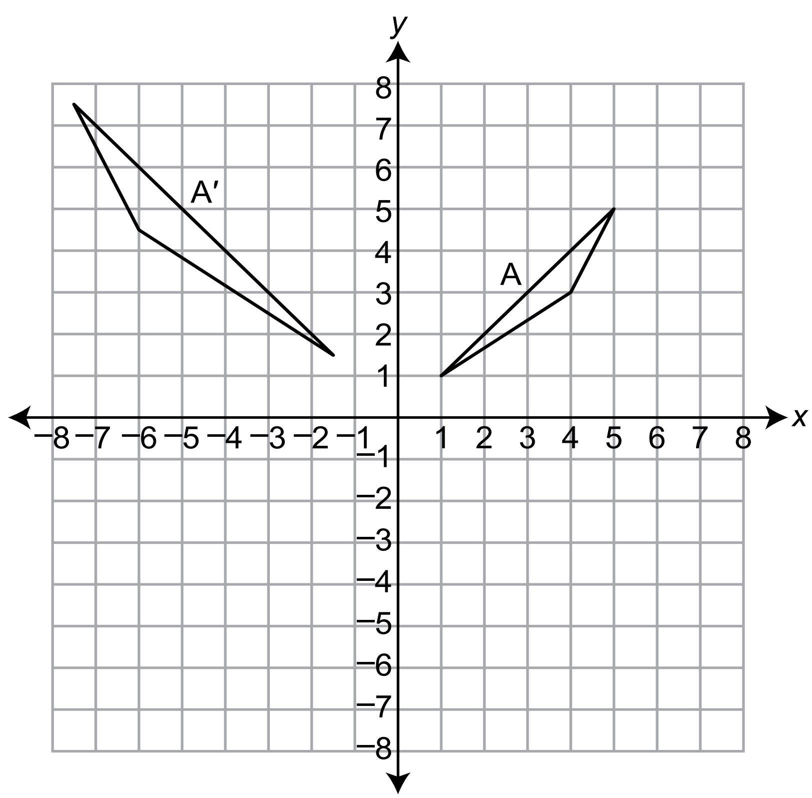


Then, measure the angles of the pre-image and the image to confirm that they are congruent. Measure the side lengths and notice that the side lengths of the image are 2 times as long as the side lengths of the pre-image. Since the angles are equal and the side lengths are proportional, it can be determined that the quadrilaterals are similar to one another.

How can the sequence of transformations that shows similarity between figures be determined?

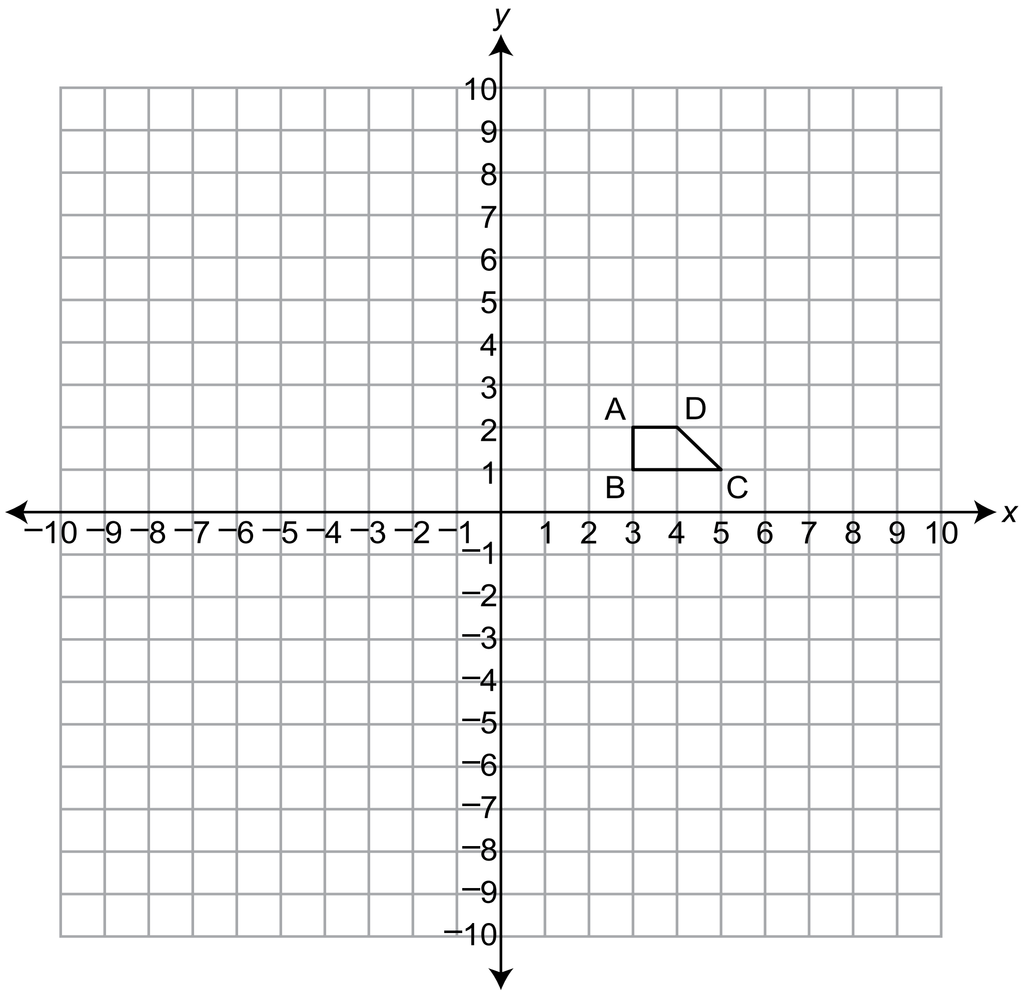
M.P.1. Make sense of problems and persevere in solving them. Look for a sequence of transformations from one figure to the next. For example, if an image has line segments that are all 10 units long and the pre-image has corresponding line segments that are 2 units long, the dilation was by a factor of 5. If the image and pre-image have corresponding angles of equal measure and the corresponding sides are flipped, then the sequence of transformations includes a reflection. Additionally, if the pre-image and the image have the same side lengths and angle measures, but the orientation of the figure is different in such a way that the figure has been moved around a fixed point, then the sequence of transformations includes a rotation.

- Ask students to determine whether a dilation, a reflection, or a combination of the two, has been used to transform the pre-image into the image. For example, give students the pre-image triangle A and the image triangle A' to analyze.

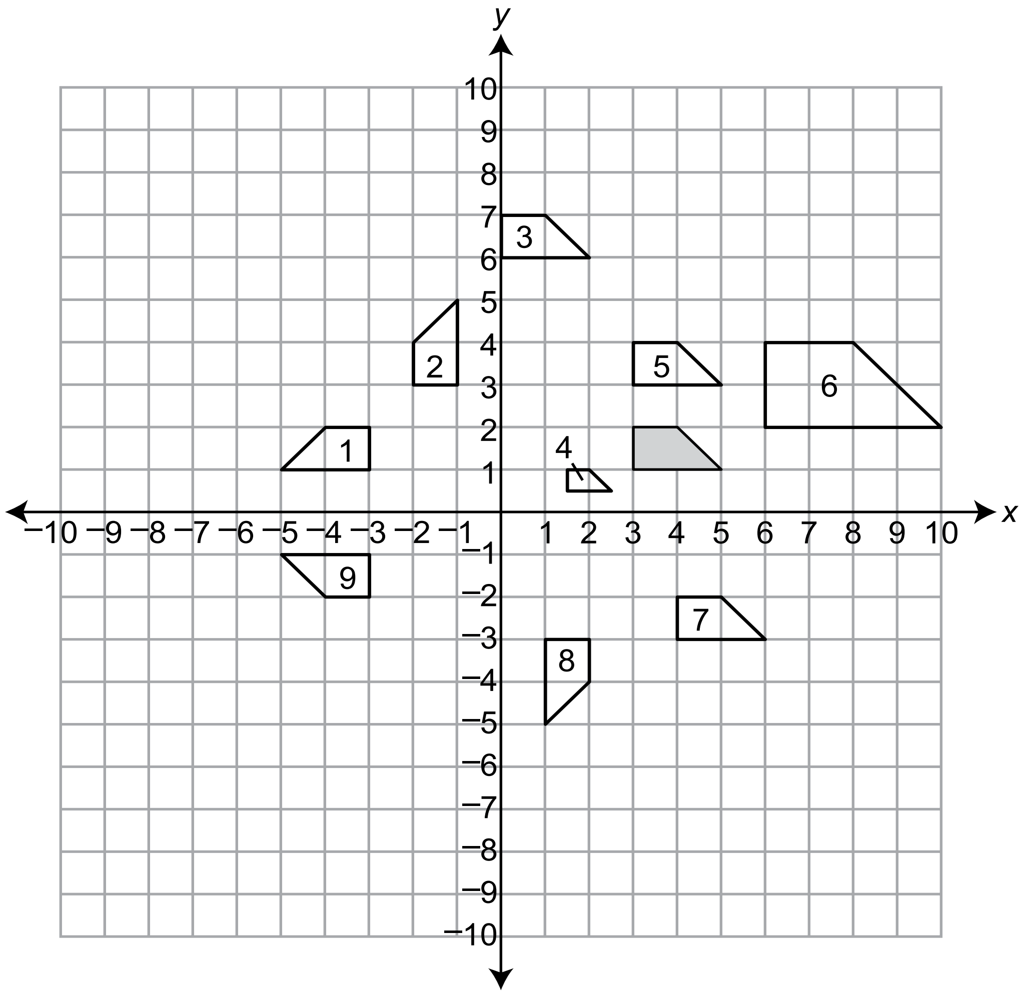


Notice that if A' were slightly smaller, it would appear to be a mirror image of A across the *y*-axis. A mirror image would indicate that a reflection was used to transform triangle A, but in this case the shapes are not the same size. Note that a dilation has been performed on triangle A, and then measure the side lengths to determine the exact factor of dilation is 1.5.

- Ask students to determine whether a rotation or translation has been used to transform from the pre-image to the image. For example, give students a pre-image, as shown.



Then, give students a coordinate grid with several transformations of the pre-image, as shown.



Explain that each image has gone through only one transformation, meaning none of the images are a combination of two transformations, such as reflection plus rotation. Then, use a table to categorize the transformations into dilations, reflections, rotations, or translations of the pre-image. The dilations have increased or decreased in size. The reflected shapes have a flipped image. The rotated shapes have an orientation of the shape that is different in such a way that the shape has been moved around the fixed point of the origin. The translations have the same size, shape, and orientation, but are moved up, down, right, or left of the pre-image. Also, use the table to explain the specific reasoning for each transformation, as shown.

Dilations	Reflections	Rotations	Translations
4 and 6 (different size than original, but the same angles, so they are dilations of the original)	1 (shape is a mirror image of the original, so it is a reflection)	2, 8, and 9 (figures have been rotated about the origin)	3, 5, and 7 (orientation is the same for these figures)

Key Academic Terms:

rotation, reflection, translation, transformation, rigid, image, pre-image, congruent, preserve, dilation, similar, corresponding sides, corresponding angles, exhibit, map, scale factor, scale up, scale down, proportional, ordered pair

Additional Resources:

- Activity: Creating similar triangles
- Activity: Different Areas?
- Video: Congruent vs similar triangles
- Lesson: Introduction to congruence and similarity through transformations
- Lesson: Dilations 1
- Video: Human tree: dilations
- Video: Rotation and dilation
- Video: Understanding dilations

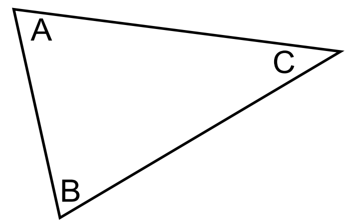
Geometry and Measurement
Analyze parallel lines cut by a transversal.
25. Analyze and apply properties of parallel lines cut by a transversal to determine missing angle measures. a. Use informal arguments to establish that the sum of the interior angles of a triangle is 180 degrees.

Guiding Questions with Connections to Mathematical Practices:

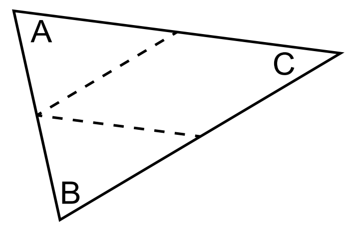
Why is the sum of the angles of a triangle 180°?

M.P.7. Look for and make use of structure. Demonstrate that the angle measures of triangles always sum up to 180° in a variety of ways. For example, show that a triangle with angle measures of 80°, 40°, and 60° has the angle sum of 180° by cutting the angles out and placing them together to show that they make a line. Additionally, use symmetry of a square to show that when a line is drawn through the diagonal of the square, the two triangles that result have angles that are equivalent, and thus, since the angle sum of the square is 360°, the angle sum of each of the two triangles created is 180°.

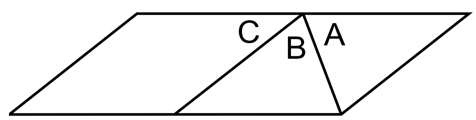
- Ask students to cut the angles out of a triangle and arrange them to show that the sum of the angles is 180°. For example, give students triangle ABC where the angle measures are unknown.



Then, cut on the dashed lines to create three sections of the triangle, as shown.

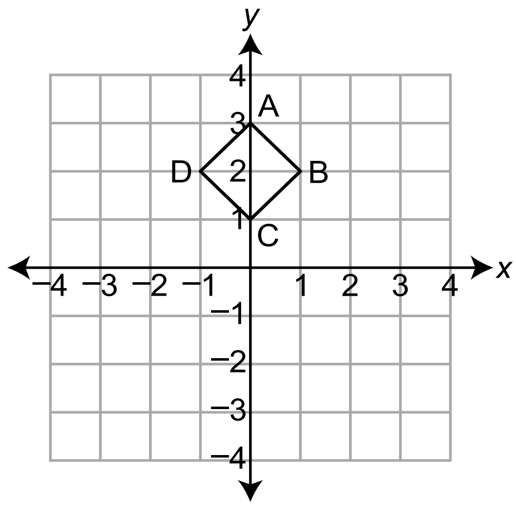


After cutting the triangle, arrange the three sections in such a way that angles A, B, and C are next to each other, as shown.



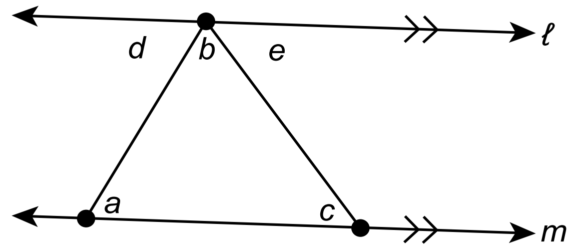
Notice that angles A, B, and C create a straight line (top side of figure), which is 180°, and therefore the sum of the angles in the triangle is 180°.

- Ask students to use symmetry to verify that the sum of angles in a triangle is 180°. For example, consider a square drawn on a coordinate grid on a piece of paper, as shown.



Fold the square along the y-axis, and note that the two triangles that result from folding are mirror images of each other across the y-axis. Triangle ADC has the same angle measures and side lengths as triangle ABC. Therefore, the sum of the angles in each triangle must be the same. Square ABCD has 4 right angles for a total of 360°. Since the sum of the angles in the two triangles must be the same, they each have half the total degrees present in the square, and therefore each triangle has angles that total 180°.

- Ask students to use parallel lines and transversals to prove that the sum of the angles in a triangle is 180°. For example, give students the figure shown.

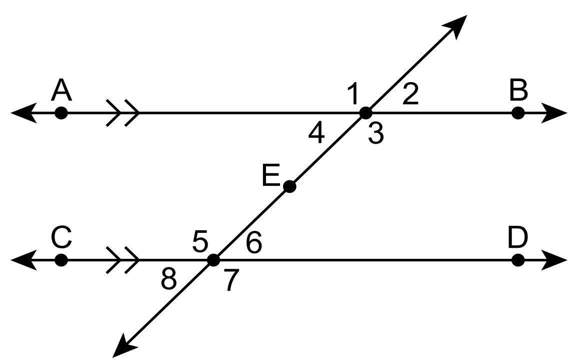


Given that lines *l* and *m* are parallel, students use congruent alternate interior angles to reason that the measure of angle *a* is equal to the measure of angle *d*, and the measure of angle *c* is equal to the measure of angle *e*. Since the measures of angles *d*, *b*, and *e* make a straight line, their sum is 180°. Using substitution for angles *d* and *e*, the measures of angles *a*, *b*, and *c* also sum to 180°.

How are the angles created by a transversal through parallel lines related?

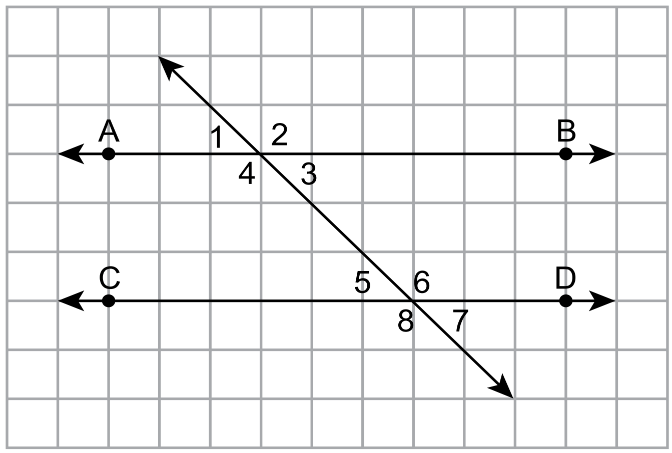
M.P.3. Construct viable arguments and critique the reasoning of others. Explain the angles of equal measure and the supplementary angles formed with a transversal through parallel lines. Use a transversal to informally prove whether lines are parallel. For example, use rotations to show that alternate interior angles have equal measure. Additionally, use translations to show that same-side interior angles are supplementary.

- Ask students to use rotations to show that alternate interior angles have the same measure. For example, give students a diagram of two parallel lines with a transversal running through them, as shown.



Ask students to draw the parallel lines and transversal on a piece of tracing paper and also label the angles of the original figure on the tracing paper. Then, rotate the diagram on the tracing paper 180° about point E so that line CD is now on top of line AB. Upon rotation, notice that angles 3 and 5 have equal measure. Angle 3 and angle 5 are on opposite sides of the transversal, making them alternate interior angles. Angle 4 and angle 6 are also alternate interior angles, and by rotation of the tracing paper, students should note that the measures of angles 4 and 6 are equal. Therefore, alternate interior angles have equal measure.

- Ask students to use transformations to show that corresponding angles are congruent and same-side interior angles are supplementary. For example, give students a diagram of two parallel lines with a transversal running through them, as shown.



Ask students to draw the parallel lines and transversal on a piece of tracing paper and also label the angles of the original figure on the tracing paper. Then, translate the tracing paper by shifting it up 3 units and left 3 units so that line CD on the tracing paper is on top of line AB on the original diagram. Notice that on the original diagram, angles 2 and 3 create a straight line, and therefore the sum of angle 2 and angle 3 is 180°. Upon translation of the tracing paper, notice that the measure of angle 2 is the same as the measure of angle 6. This is because corresponding angles are congruent. Since the measures of angle 2 and angle 3 add up to 180°, and the measure of angle 2 is the same as angle 6, then the measures of angles 3 and 6 also add up to 180°. Therefore, the same-side interior angles, angle 3 and angle 6, are supplementary.

Key Academic Terms:

triangle, angle measure, parallel lines, transversal, similarity, corresponding, alternate interior angles, alternate exterior angles, supplementary angles, vertical angles, triangle sum theorem, map

Additional Resources:

- Lesson: Lines and angles | Targeted math instruction
- Activity: Find the missing angle
- Activity: A triangle's interior angles
- Activity: Rigid motions and congruent angles
- Lesson: EngageNY Resources (Select *Mathematics Curriculum Modules* and go to geometry, module 2, student materials, lesson 15)

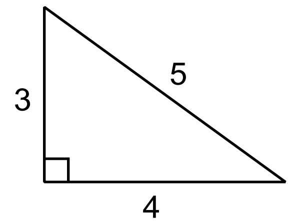
Geometry and Measurement
Understand and apply the Pythagorean Theorem.
26. Informally justify the Pythagorean Theorem and its converse.

Guiding Questions with Connections to Mathematical Practices:

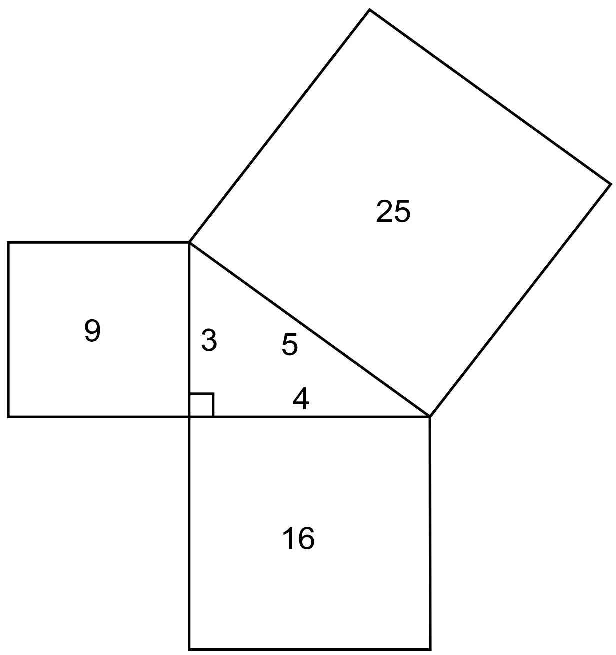
How can prior knowledge of triangles and squares be used to analyze and justify the Pythagorean Theorem?

M.P.4. Model with mathematics. Use the side lengths and areas of squares to demonstrate the Pythagorean Theorem with corresponding right triangles. For example, create squares with the same side lengths as each side of a right triangle, and use the areas of the squares to form ideas about the relationship between the legs and hypotenuse of the right triangle. Additionally, create a square with side lengths ($a + b$), and compute the area of the square in two different ways to determine the relationship between the legs and hypotenuse of the right triangle.

- Ask students to create squares with the same side lengths as each side of a right triangle and use the areas of the squares to determine the relationship between the legs and hypotenuse of the right triangle. For example, begin with a right triangle where the side lengths are labeled, as shown.



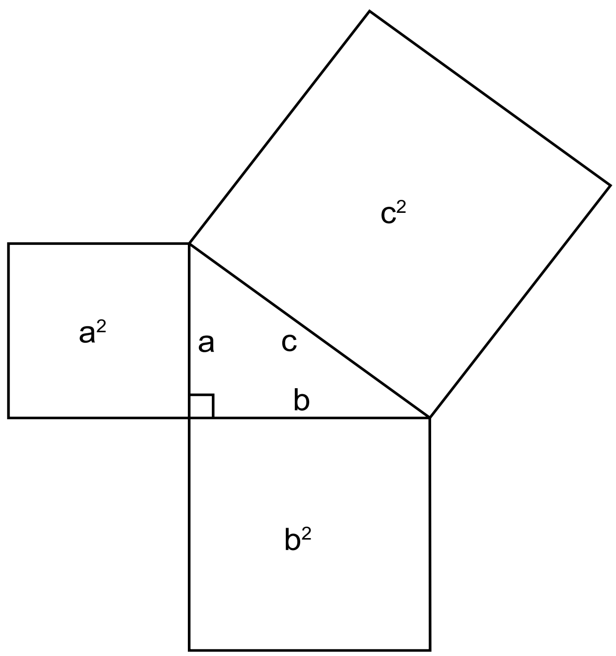
Then, create three squares, for which the side length of each square is determined by the side lengths of the right triangle. Place the squares adjacent to the corresponding side of the triangle, calculate the area of each square, and label the interior of each square with the area, as shown.



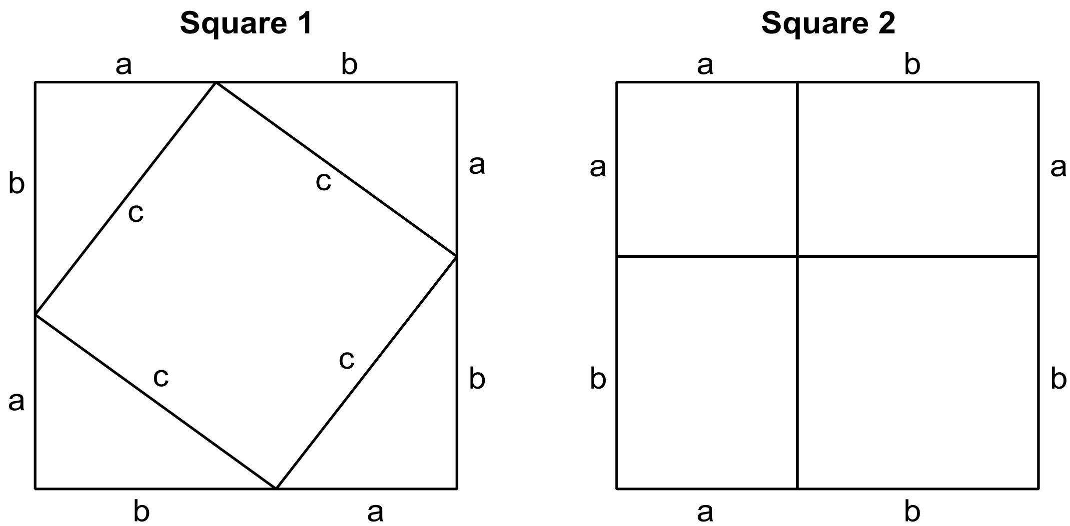
Analyze the relationship between the areas of the squares. Notice that the sum of the areas of the squares formed by the two legs of the triangle (9 square units + 16 square units) is the same as the area of the square formed by the hypotenuse of the triangle (25 square units). Have students repeat this process using other right triangles and complete a chart as shown.

Triangle Side Lengths	Longest Side Squared		Sum of the Squares of the Shorter Sides
3, 4, 5	$5^2 = 25$	=	$3^2 + 4^2 = 9 + 16 = 25$
5, 12, 13	$13^2 = 169$	=	$5^2 + 12^2 = 25 + 144 = 169$
6, 8, 10	$10^2 = 100$	=	$6^2 + 8^2 = 36 + 64 = 100$

This rule can be generalized to apply to any right triangle, with legs labeled *a* and *b* and the hypotenuse labeled *c*, to show that $a^2 + b^2 = c^2$.



- Ask students to examine two different squares, each with side length $(a + b)$, to determine a relationship between the legs and the hypotenuse of a right triangle. For example, consider two congruent squares with side length $(a + b)$ that are divided into sections, as shown in the following diagram.



The area of each square can be calculated.

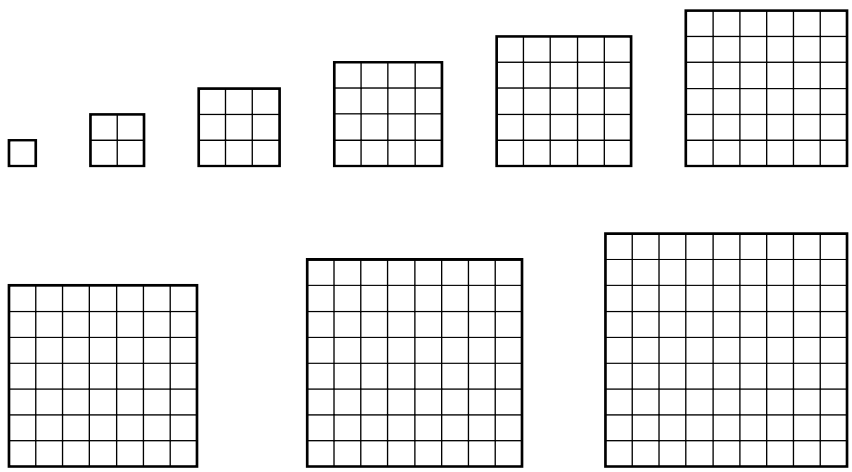
- Square 1 Area: $4(\frac{1}{2}ab) + c^2$
- Square 2 Area: $a^2 + 2ab + b^2$

Since the area of Square 1 and the area of Square 2 are equivalent, an equation can be written equating the two areas: $4(\frac{1}{2}ab) + c^2 = a^2 + 2ab + b^2$. After subtracting $2ab$ from each side of the equation, the result shows $c^2 = a^2 + b^2$ as the relationship between the side lengths and the hypotenuse of the right triangle.

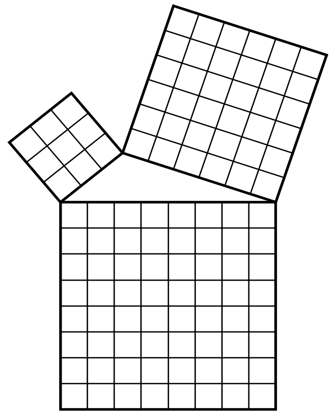
How can the converse of the Pythagorean Theorem be used to determine the type of triangle given its side lengths?

M.P.7. Look for and make use of structure. Use the square of the longest side of a triangle and the sum of the squares of the two other sides to determine whether the triangle is a right, obtuse, or acute triangle. For example, a triangle with side lengths of 10, 24, and 26 must be a right triangle because $26^2 = 10^2 + 24^2$. By contrast, a triangle with side lengths of 3, 5, and 6 is not a right triangle, because $6^2 > 3^2 + 5^2$, which means the triangle is obtuse, and a triangle with side lengths of 5, 9, and 10 is acute because $10^2 < 5^2 + 9^2$. Additionally, use Pythagorean triples to determine whether a triangle is a right triangle.

- Ask students to use the side lengths of a triangle to determine whether the triangle is right, acute, or obtuse. For example, cut out 9 squares using grid paper, with side lengths 1 through 9.



Start by asking students what kind of triangle would be created with side lengths 3, 6, and 8. Use the cut-out squares with side lengths 3, 6, and 8 to create a triangle.

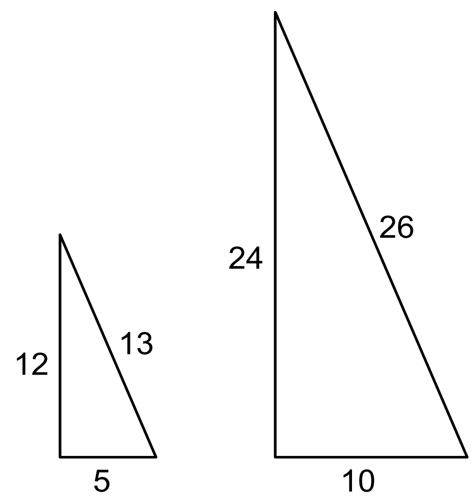


When comparing the square of the longest side to the sum of the squares of the smaller two sides, the inequality representing the squares of the triangle's side lengths is $8^2 > 3^2 + 6^2$. Also observe that the triangle is obtuse. Continue the process of creating triangles using the squares cut out from the coordinate grid, determine the type of triangle and determine the correct equation or inequality comparing the longest side squared to the sum of the squares of the shortest two sides.

Triangle Side Lengths	**Longest Side Squared**	**>, <, or =**	**Sum of the Squares of the Shorter Sides**	**Acute, Obtuse, or Right**
3, 6, 8	8^2	>	$3^2 + 6^2$	obtuse
7, 8, 9	9^2	<	$7^2 + 8^2$	acute
3, 4, 5	5^2	=	$3^2 + 4^2$	right

Test several more triangles with various side lengths until students feel comfortable determining a pattern. Notice that when $c^2 = a^2 + b^2$, the triangle is always a right triangle, when $c^2 > a^2 + b^2$, the triangle is always obtuse, and when $c^2 < a^2 + b^2$, the triangle is always acute.

- Ask students to investigate Pythagorean triples, and use them to determine whether a given triangle is a right triangle. For example, start by defining Pythagorean triples to students as a set of positive integers, a, b, and c, for which $a^2 + b^2 = c^2$. When those integers are the side lengths of a triangle, the triangle is a right triangle. For example, $3^2 + 4^2 = 5^2$, so 3-4-5 is a Pythagorean triple. This means a triangle with side lengths of 3, 4, and 5 is a right triangle because the side lengths are a Pythagorean triple. Additionally, 5, 12, 13 is also considered a Pythagorean triple since $5^2 + 12^2 = 13^2$, and therefore a triangle with side lengths 5, 12, and 13 is a right triangle. Students should also recall that multiplying all side lengths of a triangle by the same constant creates similar triangles. Therefore, when the side lengths of a right triangle created by a Pythagorean triple are multiplied by a constant, a new Pythagorean triple is produced and the result is also a right triangle. For instance, consider the two triangles shown.



Recognize 5-12-13 as a Pythagorean triple. Thus, these side lengths produce a right triangle, and it can be verified because $5^2 + 12^2 = 13^2$. Students should also notice that side lengths of 10, 24, and 26 are double the side lengths of the 5-12-13 triangle. Therefore, 10-24-26 is also a Pythagorean triple, which produces a right triangle as well.

Key Academic Terms:

right triangle, Pythagorean Theorem, leg, hypotenuse, converse, sides, obtuse triangle, acute triangle, proof, equation, square root, inverse operation, side length, line segment

Additional Resources:

- Article: Teaching the Pythagorean Theorem proof through discovery
- Article: Hands-on explorations of the Pythagorean Theorem
- Activity: Pythagorean Theorem trapezoid proof by President Garfield

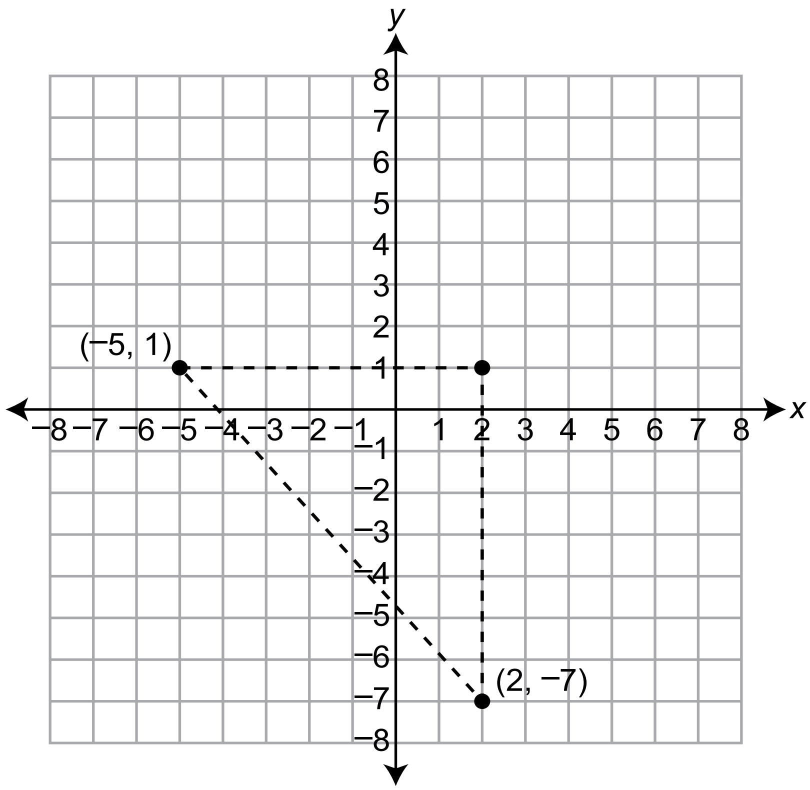
Geometry and Measurement
Understand and apply the Pythagorean Theorem.
27. Apply the Pythagorean Theorem to find the distance between two points in a coordinate plane.

Guiding Questions with Connections to Mathematical Practices:

How can the Pythagorean Theorem be used to find the distance between points in a coordinate plane?

M.P.7. Look for and make use of structure. Find the right triangle that corresponds to any two points on a coordinate grid by drawing a line segment between them, and then determine the horizontal and vertical legs based on the two points that meet at a right angle; the distance between the two points is the hypotenuse of the right triangle. For example, the points (1, 1) and (5, 6) make the hypotenuse, c, of a triangle that has its right-angle vertex at (5, 1). As such, the right triangle has legs of length 4 and length 5, and so the distance between (1, 1) and (5, 6) will be the value that solves the equation $c^2 = 4^2 + 5^2$. Additionally, if the center of a circle is at (2, 6) and the circle passes through the point (−3, 4), the length of the radius can be determined by finding the square root of the sums of the squares of the horizontal and vertical distances between the two points.

- Ask students to draw a right triangle onto a coordinate grid as a tool for finding the distance between two points. For example, a student could draw a right triangle on the coordinate grid with the hypotenuse connecting the points (2, −7) and (−5, 1), as shown. Therefore, the distance between these points can be calculated by determining the length of the hypotenuse of the right triangle by solving the equation $c^2 = 7^2 + 8^2$.



- Ask students to write an expression that can be used to determine the distance between two points on the coordinate plane. For example, given the points (−1, 6) and (2, 4), the distance between them can be calculated as $\sqrt{(-1 - 2)^2 + (6 - 4)^2}$. Note that while formal introduction to the distance formula is not part of Grade 8 standards, this equation may be derived using the Pythagorean Theorem.

Key Academic Terms:

right triangle, Pythagorean Theorem, legs, hypotenuse, distance, coordinate system, line segment, vertex, equation, square root, ordered pair, side length

Additional Resources:

- Video: Applying the Pythagorean Theorem
- Activity: Finding isosceles triangles
- Activity: Finding the distance between points

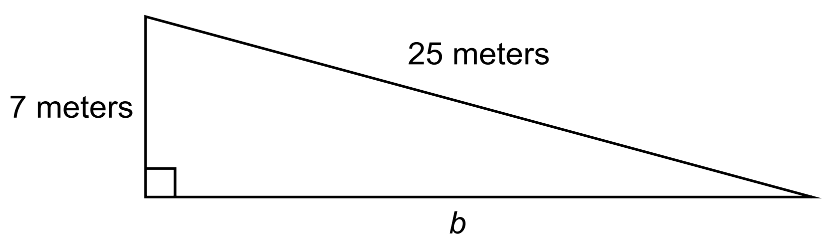
Geometry and Measurement
Understand and apply the Pythagorean Theorem.
28. Apply the Pythagorean Theorem to determine unknown side lengths of right triangles, including real-world applications.

Guiding Questions with Connections to Mathematical Practices:

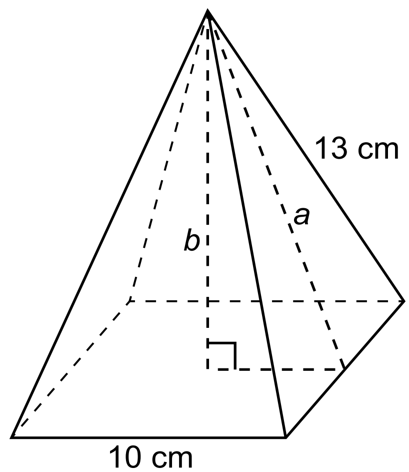
How can the Pythagorean Theorem be used to find unknown side lengths in right triangles?

M.P.6. Attend to precision. Substitute the known side lengths into the equation $a^2 + b^2 = c^2$ and then solve the equation for the unknown side length in two and three dimensions. For example, if the structure underneath a walking bridge is made up of right triangles that each have a hypotenuse that is 12 feet long and one leg that is 9 feet long, find the length of the missing leg by solving the equation $a^2 + 9^2 = 12^2$ for a. Additionally, if two legs of a right triangle measure 5 units and 9 units, the length of the hypotenuse, c units, may be determined by solving the equation $5^2 + 9^2 = c^2$.

- Ask students to determine the length of the hypotenuse when given the leg lengths in a right triangle. For example, given leg lengths of 8 units and 3 units, solve for c in the equation $c^2 = 8^2 + 3^2$ to find a hypotenuse measuring $\sqrt{73}$, or approximately 8.544 units.
- Ask students to determine the length of a leg in a right triangle when given the measurements of the hypotenuse and the other leg. For example, using the triangle shown, determine the length of the longer leg by solving the equation $25^2 = 7^2 + b^2$ to get a length of 24 meters.



- Ask students to determine measurements in a right rectangular pyramid using the Pythagorean Theorem. For example, the pyramid shown has an edge measuring 13 centimeters and a square base with side lengths of 10 centimeters. Solve for a in the equation $a^2 + 5^2 = 13^2$ (where a is the slant height, 5 is half of a side length of the square base, and 13 is an edge of the square pyramid), which yields a slant height of 12 centimeters. Then, use the slant height of 12 to determine the height of the pyramid by noticing the dashed right triangle in the figure and solving for b in the equation $5^2 + b^2 = 12^2$. Explain to students that a, the slant height, is the hypotenuse of the dashed right triangle in the figure.



Key Academic Terms:

right triangle, Pythagorean Theorem, leg, hypotenuse, dimensions, unit, equation, square root, slant height

Additional Resources:

- Video: [Applying the Pythagorean Theorem]
- Lesson: [Exponents and radicals | Targeted math instruction]
- Activity: [Areas of geometric shapes with the same perimeter]

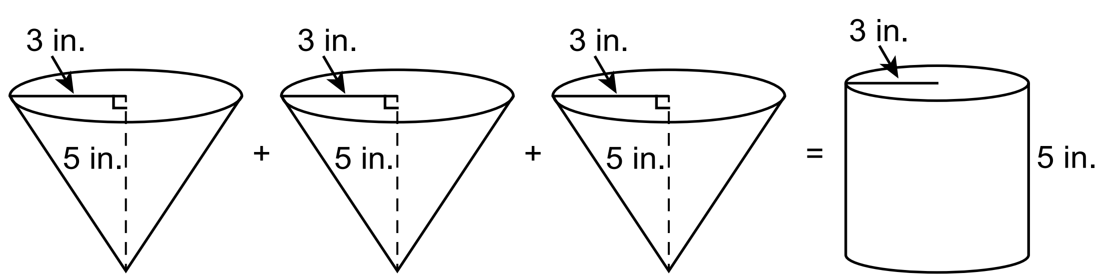
Geometry and Measurement
Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
29. Informally derive the formulas for the volume of cones and spheres by experimentally comparing the volumes of cones and spheres with the same radius and height to a cylinder with the same dimensions.

Guiding Questions with Connections to Mathematical Practices:

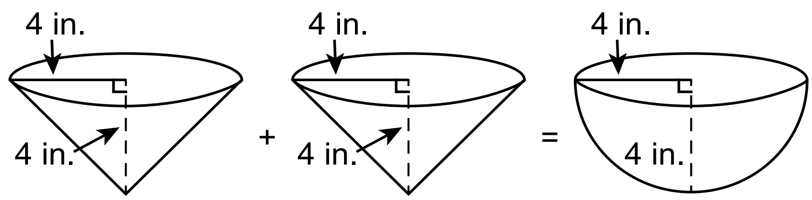
How are the volumes of cones and cylinders connected to each other?

M.P.7. Look for and make use of structure. Explore volume with manipulatives and formulas to find the pattern of multiplying the area of a circle by the constant appropriate to the given figure. For example, a cone has the volume of the corresponding cylinder (i.e., with the same base and height) multiplied by one-third. Additionally, a sphere has the volume of the corresponding cylinder (i.e., with the same circular cross section and height) multiplied by two-thirds.

- Ask students to investigate the volume relationship of cylinders and cones using manipulatives, as shown. For example, find the volume of water that an empty cylinder with a circular base area of 9π square inches and a height of 5 inches can hold. Reason that 3 equally sized cones with the same base area and height (or the same cone 3 times) can be filled with water and emptied into the cylinder, thereby filling it completely. Therefore, the volume of the cylinder shown is 45π cubic inches and the volume of each cone is 15π cubic inches.



- Ask students to investigate the volume relationship of spheres and cones using manipulatives, as shown. For example, find the volume of an empty hemispherical bowl with a circular cross section area of 16π square inches and a height of 4 inches. Reason that the hemisphere can be filled with water by pouring in the water from 2 cones where both the radius and height of each cone are equal to the radius of the hemisphere.

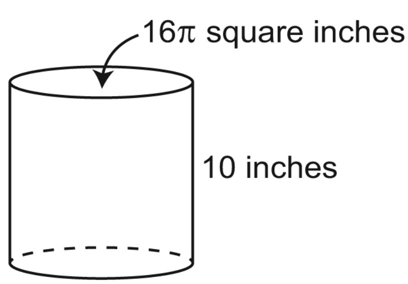


An additional connecting activity can be done using an empty cylinder, an empty cone, and an empty hemisphere where both the radius and height of the cone and cylinder are equal to the radius of the hemisphere. Students can be led to discover that the volume of the cone plus the volume of the hemisphere equals the volume of the cylinder.

How is the volume of a cylinder connected to the area of circles?

M.P.7. Look for and make use of structure. Demonstrate the volume of a cone, cylinder, or sphere as the area of a circle, a plane section of the figure, multiplied by another dimension of the figure. For example, recognize the formula for the volume of a cylinder as the area of a circular base multiplied by its height. Additionally, the volume of a cone is one-third the area of its circular base multiplied by its height, and the volume of a sphere is two-thirds of the area of its circular cross section through the center multiplied by its height.

- Ask students to determine the volume of a cylinder by finding the product of the area of the circular base and the height. For example, a cylinder of height 10 inches is sliced horizontally such that the area of the circular cross section is 16π, meaning the volume of the cylinder can be found using the product of 10 and 16π, which is 160π cubic inches, or about 502.65 cubic inches.



Key Academic Terms:

volume, cone, cylinder, sphere, pi, constant, radius, diameter, irrational number, formula, equation, variable

Additional Resources:

- Activity: Comparing snow cones
- Activity: Shipping rolled oats
- Tutorial: Cone vs sphere vs cylinder
- Article: 12 engaging ways to practice volume of cylinders, cones, and spheres

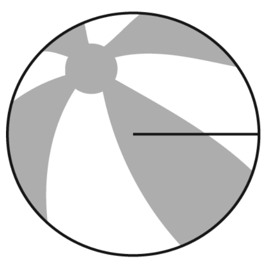
Geometry and Measurement
Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. *Note: Students must select and use the appropriate unit for the attribute being measured when determining length, area, angle, time, or volume.*
30. Use formulas to calculate the volumes of three-dimensional figures (cylinders, cones, and spheres) to solve real-world problems.

Guiding Questions with Connections to Mathematical Practices:

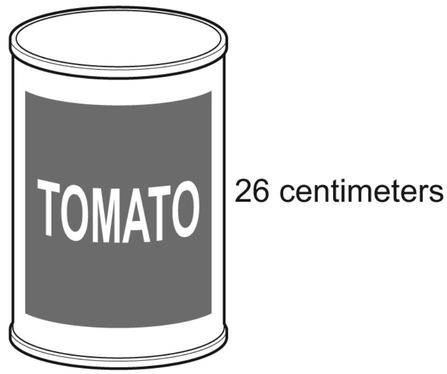
How can real-world and mathematical problems be solved using the volumes of cones, cylinders, and spheres?

M.P.6. Attend to precision. Represent the unknown in a problem involving the volume of a cone, cylinder, or sphere with a variable and use the formula to solve for the unknown. For example, given the height and radius of a cylinder-shaped food container, use the formula to find the volume of food the container can hold. Additionally, given the radius and volume of a cylinder-shaped food container, use the formula to write and solve an equation for the height.

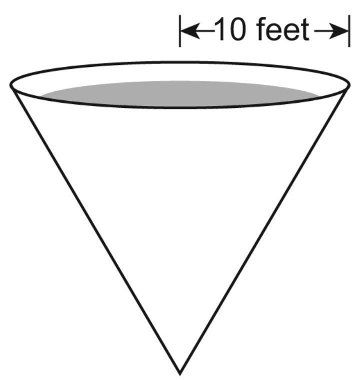
- Ask students to calculate volume using the appropriate formula. For example, find the volume of a beach ball with a radius of 15 inches by using $V = \frac{4}{3}\pi(15^3)$.



- Ask students to calculate the radius of a three-dimensional figure using known volume. For example, if a cylinder-shaped food container holds 4,000 cubic centimeters of product and has a height of 26 centimeters, determine that the radius of the cylinder, to the nearest tenth of a centimeter, is 7.0 centimeters by solving the equation $26\pi r^2 = 4{,}000$ for r.



- Ask students to calculate the height of a three-dimensional figure using known volume. For example, if a conical reservoir with a base radius of 10 feet is filled with 3,000 cubic feet of grain, determine that the height of the reservoir, to the nearest tenth of a foot, is 28.7 feet by solving the equation $\frac{1}{3}\pi(10^2)h = 3{,}000$ for h.



Key Academic Terms:

volume, cone, cylinder, sphere, pi, constant, radius, diameter, formula, equation, variable

Additional Resources:

- Tutorials: Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.
- Video: Real-life math | Civil Engineer
- Activity: Glasses